Misiurewicz point patterns generation in one-dimensional quadratic maps

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Abstract

In a family of one-dimensional quadratic maps, Misiurewicz points are unstable and the orbits of such a points are repulsive. On the contrary, the orbits of superstable periodic points are attractive. Here we study the patterns of the symbolic sequences of both Misiurewicz and superstable periodic points, and show that a Misiurewicz point pattern can be obtained as the limit of the sum of a superstable periodic orbit pattern plus itself or some of its heredity transmitters repeated an infinite number of times. Inversely, when a Misiurewicz point pattern is given, we also show that it is possible to find both the superstable periodic orbit pattern and the heredity transmitters that generate such a pattern.

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1. Introduction

In 1947, Ulam and von Neumann [1] used the one-dimensional (1D) quadratic map $x_{k+1} = 4x_k(1 - x_k)$ as a random number generator. Years later, Myrberg studied the iterates of the family of 1D quadratic maps $y_{k+1} = y_k^2 - p$ when the parameter p is varied [2, 3], and he saw the existence of same-period superstable orbits for different parameter values. Myrberg showed the difference between these orbits using sequences of signs "$+$" (when the iterate y_k was $y_k > 0$) and "$-$" (when the iterate y_k was $y_k < 0$). Years later, Metropolis, Stein and Stein (MSS) also used symbolic sequences (that they named patterns) for differentiating the orbits in the logistic difference equation, normally called the logistic map,

$$x_{k+1} = \lambda x_k(1 - x_k) \qquad (1)$$

and others families of 1D unimodal maps [4]. They used the letter "R" (when the iterate x_k was $x_k > x_0$, where $x_0 = 0.5$ is the map critical point) and the letter "L" (when the iterate x_k was $x_k < x_0$). In this way, they ordered a lot of superstable periodic orbits patterns in the parameter straight line. Later, May showed in a review article that a simple mathematical model as the family of maps of Eq. (1), can exhibit a surprising array of dynamical behavior [5].

Dynamical systems theory describes the time evolution of a system as a trajectory, or an orbit, in a phase space of the system's possible states. For dissipative systems with a chaotic attractor that appears locally two-dimensional, a cross section through the attractor intersects it in some curve [6]. One can then consider the dynamics as a map from this cross section onto itself: this map is called the Poincaré map. In this sense, discrete time maps summarize the dynamics underlying chaotic behavior found in higher dimensional systems. From their simplicity, 1D maps have developed as prototypical models in the study of chaotic dynamics [6]. For dissipative dynamical systems, such as discrete mappings and ordinary and partial differential equations that exhibit cascading bifurcations, the dynamics can be described in practice by a 1D map with a single smooth maximum, as a 1D quadratic map [6]. In fact, recently, Beck has shown that the stroboscopic dynamics of a kicked charged particle moving in a double-well potential and a time-dependent magnetic field reduces, in certain cases, to the complex logistic map [7]. This result shows the interest of studying quadratic maps to better understand some physical dissipative dynamical systems.

As is known, in a parameter dependent family of 1D quadratic maps they are both periodic points and periodic points with preperiod. Same-period periodic points are jointed in intervals but periodic points with preperiod, named Misiurewicz points, are isolated in the parameter straight line. As it is known, periodic points have negative Lyapunov exponent and periodic points with preperiod have positive Lyapunov exponent [8]. In the practical, nonetheless, the Lyapunov exponent of a lot of periodic points result positive when they are calculated by a computer with finite machine accuracy, and they look like chaotic points. Misiurewicz points are important because they are very numerous and they are the only strictly chaotic points in 1D quadratic maps [8]. We became aware of the interest of Misiurewicz points when we wanted to study the superstable periodic orbits ordering in the family of 1D quadratic maps

$$x_{k+1} = x_k^2 + c, \qquad (2)$$

and then we saw the importance of Misiurewicz points in that such ordering [9]. We name these points as $M_{n,p}$, where *n* is the preperiod and *p* is the period of the corresponding orbit. Misiurewicz points in a family of 1D quadratic maps had already been treated by other authors [10-13]. We published an eight-pages table with both parameter values and symbolic sequences of a lot of Misiurewicz points in the family of maps of Eq. (2) [14, 15]. Now we show a model for the generation of symbolic sequences of Misiurewicz points in a family of 1D quadratic maps starting from the symbolic sequences of its periodic orbits.

As done in previous articles [9, 14-16], in order to study the ordering of superstable periodic orbits in the family of 1D quadratic maps of Eq. (2) we draw the antenna of the Mandelbrot set [17] with the escape line method [18]. In this way we can see, with the naked eye, the period of each one of the midgets of the Mandelbrot set antenna. Next, we situate a Misiurewicz point in the limit of a series of midgets. As it is known, the ordering of both the midgets in the Mandelbrot set antenna and the periodic orbits in the family of maps of Eq. (2) is the same. In this work we shall depict figures with two different parts: in the upper part we shall show the generation model of the Misiurewicz point in the family of 1D quadratic maps of Eq. (2), and in the lower part we shall show the corresponding zone of the Mandelbrot set antenna represented by means of the escape lines method.

The symbolic dynamics provides almost the only rigorous way to understand global systematic of periodic and, especially, chaotic motion in dynamical systems [19]. The applied symbolic dynamics originated from the papers of Myrberg [3] and Metropolis, Stein and Stein [4] and is becoming a practical tool for the study of the ordering of periodic orbits in families of 1D unimodal maps [20]. As is known, the period-three superstable orbit in the family of maps $y_{k+1} = y_k^2 - p$ occurs for $p = 1.754877...$ [3]. In fact, if we iterate the map corresponding to this parameter value from the initial point $y_0 = 0$ (the critical point of the map), we obtain the period-three superstable orbit $\{0,\ -1.75\ldots,\ 1.32\ldots\}$. The Myrberg symbolic sequence of this orbit is only a sign "+" corresponding to the second iterate 1.32…, because the initial point (always 0) and the first iterate (always a negative value) are not take into account. Note that all the Myrberg symbolic sequences begin whit the sign "+" because the second iterate always is a positive value. So, the Myrberg symbolic sequence of a period-*p* superstable orbit has $p-2$ signs. On the other hand, the period-three superstable orbit in the family of maps of Eq. (1) occurs for $\lambda = 3.8318741\ldots$ [4], and the orbit from the initial point $x_0 = 0.5$ (the critical point) is $\{0.5,\ 0.95\ldots,\ 0.15\ldots\}$. The MSS pattern of this orbit is "RL" because the initial point (always 0.5) is not take into account. Note that all the MSS pattern begin with the letters RL and the MSS pattern of a period-*p* superstable orbit has $p-1$ letters.

However, a pattern of a period-*p* superstable orbit with a number of symbols less than *p* can be misleading. To avoid this, we write the symbolic sequence of such an orbit with *p* letters R's or L's, according to other authors [19, 21, 22]. Moreover, as is known, there are two types of 1D quadratic maps: rightward maps (R maps) with negative second derivative, and leftward maps (L maps) with positive second derivative [23]. Then, the Eq. (1) corresponds to a family of rightward maps, and the Eq. (2) corresponds to a family of leftward maps. The patterns of equivalent orbits in these two types of maps interchange the L's and R's. So, we write the pattern of the period-3 superstable orbit in the family of rightward maps of Eq. (1) as CRL (where C symbolizes the initial value $x_0 = 0.5$), and we write the pattern of the period-3 superstable orbit in the family of leftward maps of Eq. (2) as CLR (where C symbolizes the initial value $x_0 = 0$).

The pattern of a superstable periodic orbit has parity [15, 22]. The R parity (odd or even, according to the R's number) is the parity used in rightward maps. The L parity (odd or even, according to the L's number) must be used in leftward maps. So, the pattern CRL^2 in the family of rightward maps of Eq. (1) has odd R parity, and the pattern $\mathrm{CLR}^2\mathrm{L}$ in the family of leftward maps of Eq. (2) has even L parity.

In a family of 1D quadratic maps a superstable periodic orbit pattern P_2 can be added to another superstable periodic orbit pattern P_1 to obtaining a composition pattern [23]. This addition can be done in two different ways; toward the left ($P_1 \overleftarrow{+} P_2$) and toward the right ($P_1 \overrightarrow{+} P_2$). There are mnemonic rules, the leftward and the rightward rules [23], to do it. For example, let us consider the family of 1D quadratic leftward maps of Eq. (2). The composition pattern $P_1 \overrightarrow{+} P_2$ is formed by appending P_2 to P_1 and changing the C of P_2 to R (or L) if the L parity of P_1 is even (or odd). So, if $P_1 = \mathrm{CLR}$ and $P_2 = \mathrm{CL}$ we have $P_1 \overrightarrow{+} P_2 = \mathrm{CLRL}^2$.

We can add a pattern *P* to itself in the canonical direction (toward the left in L maps, or toward the right in R maps) a given number of times. In this case we obtain the harmonics of pattern *P* [4, 15]. For example, the first and second harmonics of pattern CLR in the family of leftward maps of Eq. (2) are $H^{(1)}(\mathrm{CLR}) = \mathrm{CLR} \overleftarrow{+} \mathrm{CLR} = \mathrm{CLR}^2\mathrm{LR}$ and $H^{(2)}(\mathrm{CLR}) = \mathrm{CLR} \overleftarrow{+} \mathrm{CLR} \overleftarrow{+} \mathrm{CLR} = \mathrm{CLR}^2\mathrm{LRL}^2\mathrm{R}$. We can also add a pattern *P* to itself in the anticanonical direction (toward the left in R maps, or toward the right in L maps) a given number of times. In this case we obtain the antiharmonics of pattern *P* [4, 15]. As is known, the harmonics of the pattern of a superstable periodic orbit correspond to existing patterns, but the antiharmonics not [4, 23, 24].

In a recent paper the heredity concept in a family of 1D quadratic maps has been introduced by us [25]. Given the pattern P of a superstable periodic orbit it is possible calculate the complete family tree of such a pattern. To this end, we first carry out the ancestral decomposition of P obtaining their heredity transmitters. Next, we can generate the patterns of the descendants of P by composition of P with its heredity transmitters. As can be seen later, the heredity is also present in the generation of Misiurewicz points patterns.

It is useful to extend the use of the applied symbolic dynamics to Misiurewicz points [14, 15]. We represent the pattern of the orbit of a Misiurewicz point $M_{n,p}$ with a sequence of n letters in brackets (the preperiodic part) followed by p letters without brackets (the periodic part). So, as it is easy verify by iterating, the Misiurewicz point pattern $M_{3,1}$ in the family of maps of Eq. (2), located to the parameter value $c = -1.543689\ldots$, is $(\mathrm{CLR})\mathrm{L}$.

We have tried in the article to be clear and didactic. For that, we have only shown a little of the enormous underlying fieldwork that is hidden from view, and we have only introduced such data as needed for the exposition. However, we would like to emphasize that each one of the assertions and properties introduced here is the result of a wide sampling that in most cases is not shown.

Starting from the pattern of a superstable periodic orbit of a family of 1D quadratic maps, we obtain the pattern of a Misiurewicz point by three methods. We will explain these methods for the case of a family of 1D quadratic leftward maps; the extension to a family of 1D quadratic rightward maps is immediate. We shall start from the first method, the generation of a Misiurewicz point from the composition of a pattern with itself.

2. Composition of a pattern with itself

Let P be a superstable periodic orbit pattern in a family of 1D quadratic leftward maps. We can write $P = \mathrm{C}Q$, where Q is the pattern P without the initial letter C. If we add indefinitely the pattern P to itself always toward the left [23] we have

$$P \overleftarrow{+} P \overleftarrow{+} P \overleftarrow{+} \ldots = P\left[\overleftarrow{P}\right]^{\infty} = \mathrm{C}Q\overleftarrow{\mathrm{C}}Q\overleftarrow{\mathrm{C}}Q\overleftarrow{\mathrm{C}}Q\ldots. \qquad (3)$$

Let us call Q_{e} a Q whose L parity is even, and Q_{o} a Q whose L parity is odd. There are two cases in Eq. (3). By substituting for each one of them, we obtain if $Q = Q_{\mathrm{o}} \;\Rightarrow\; P\left[\overleftarrow{P}\right]^{\infty} = \mathrm{C}Q_{\mathrm{o}}\mathrm{R}Q_{\mathrm{o}}\mathrm{L}Q_{\mathrm{o}}\mathrm{L}Q_{\mathrm{o}}\mathrm{L}Q_{\mathrm{o}}\ldots,$

if $Q = Q_{\mathrm{e}} \;\Rightarrow\; P\left[\overleftarrow{P}\right]^{\infty} = \mathrm{C}Q_{\mathrm{e}}\mathrm{L}Q_{\mathrm{e}}\mathrm{R}Q_{\mathrm{e}}\mathrm{R}Q_{\mathrm{e}}\mathrm{R}Q_{\mathrm{e}}\ldots,$

and in the two cases the pattern $P\left[\overleftarrow{P}\right]^{\infty}$ has a preperiod and it is eventually periodic, as is the pattern of a Misiurewicz point. Note that the periodic part of this pattern has even L parity because both $\mathrm{L}Q_{\mathrm{o}}$ and $\mathrm{R}Q_{\mathrm{e}}$ have even L parity, and the period of this periodic part equals to the period of pattern *P*. We can give the following property:

Property 1. Let *P* be a superstable periodic orbit pattern in a family of 1D quadratic leftward maps. If we add indefinitely the pattern *P* to itself always toward the left the pattern obtained corresponds to a Misiurewicz point.

Let us see two simple examples. First, let $P = \mathrm{CLR}$ be the pattern of the superstable period-3 orbit in the family of leftward maps of Eq. (2). This pattern correspond to the pattern of the cardioid of the period-3 midget in the Mandelbrot set antenna. If we add indefinitely the pattern *P* to itself always toward the left we have $P \overleftarrow{+} P \overleftarrow{+} P \overleftarrow{+} \ldots =$ $\mathrm{CLR} \overleftarrow{+} \mathrm{CLR} \overleftarrow{+} \mathrm{CLR} \overleftarrow{+} \ldots =$ $(\mathrm{CLR}^2)\mathrm{LRL}$. As is known, $(\mathrm{CLR}^2)\mathrm{LRL}$ is the pattern of the tip of the period-3 midget in the Mandelbrot set antenna, the Misiurewicz point $M_{4,3}^{(1)}$ located at the parameter value $c = -1.790327...$ [15, 24]. It is a crisis Misiurewicz point [26]. Second, let $P = \mathrm{CL}$ be the pattern of the superstable period-2 orbit in the period doubling cascade of the family of leftward maps of Eq. (2). This pattern correspond to the period-2 disk tangent to the main cardioid of the Mandelbrot set. If we add indefinitely the pattern *P* to itself always toward the left we have $P \overleftarrow{+} P \overleftarrow{+} P \overleftarrow{+} \ldots =$ $\mathrm{CL} \overleftarrow{+} \mathrm{CL} \overleftarrow{+} \mathrm{CL} \overleftarrow{+} \ldots = (\mathrm{CLR})\mathrm{L}$. Note that $(\mathrm{CLR})\mathrm{L}$ is the pattern of the merging point that separates the chaotic bands $\mathbf{B}_0$ and $\mathbf{B}_1$ [9]. It is the Misiurewicz point $M_{3,1}^{(1)}$ located at the parameter value $c = -1.543689...$ [15]. These two types of Misiurewicz points, tips and merging points, are always obtained by composition of a pattern with itself. Tips and merging points, that we call structural Misiurewicz points, are very important in the study of the structure of a family of 1D quadratic maps [23].

Let us see now the second method, the generation of a Misiurewicz point from the composition of a pattern with one of its heredity transmitters.

3. Composition of a pattern with one of its heredity transmitters

Let P_1 and P_2 be two superstable periodic orbits patterns in a family of 1D quadratic leftward maps. As we shall see next, if we add indefinitely the pattern P_2 to P_1 always with the same direction, toward the left or toward the right, the pattern obtained has a preperiod and it is eventually periodic but sometimes it corresponds to a Misiurewicz point and sometimes not. We can write $P_1 = \mathrm{C}Q_1$ and $P_2 = \mathrm{C}Q_2$, where Q_1 and Q_2 are P_1 and P_2 without the initial letter C. If we add indefinitely the pattern P_2 to P_1 always with a same direction, toward the left as example, we have

$$P_1 \overleftarrow{+} P_2 \overleftarrow{+} P_2 \overleftarrow{+} P_2 \overleftarrow{+} \ldots = P_1\left[\overleftarrow{P_2}\right]^{\infty} = \mathrm{C}Q_1\overleftarrow{\mathrm{C}}Q_2\overleftarrow{\mathrm{C}}Q_2\overleftarrow{\mathrm{C}}Q_2\overleftarrow{\mathrm{C}}Q_2\ldots \tag{4}$$

Let us call Q_e a Q whose L parity is even, and Q_o a Q whose L parity is odd. There are four cases in Eq. (4). By substituting for each one of them, we obtain

$$\text{if } Q_1 = Q_{1\mathrm{o}} \text{ and } Q_2 = Q_{2\mathrm{o}} \;\Rightarrow\; P_1\left[\overleftarrow{P_2}\right]^{\infty} = \mathrm{C}Q_{1\mathrm{o}}\mathrm{R}Q_{2\mathrm{o}}\mathrm{L}Q_{2\mathrm{o}}\mathrm{L}Q_{2\mathrm{o}}\mathrm{L}Q_{2\mathrm{o}}\ldots,$$

$$\text{if } Q_1 = Q_{1\mathrm{o}} \text{ and } Q_2 = Q_{2\mathrm{e}} \;\Rightarrow\; P_1\left[\overleftarrow{P_2}\right]^{\infty} = \mathrm{C}Q_{1\mathrm{o}}\mathrm{R}Q_{2\mathrm{e}}\mathrm{R}Q_{2\mathrm{e}}\mathrm{R}Q_{2\mathrm{e}}\mathrm{R}Q_{2\mathrm{e}}\ldots,$$

$$\text{if } Q_1 = Q_{1\mathrm{e}} \text{ and } Q_2 = Q_{2\mathrm{o}} \;\Rightarrow\; P_1\left[\overleftarrow{P_2}\right]^{\infty} = \mathrm{C}Q_{1\mathrm{e}}\mathrm{L}Q_{2\mathrm{o}}\mathrm{L}Q_{2\mathrm{o}}\mathrm{L}Q_{2\mathrm{o}}\mathrm{L}Q_{2\mathrm{o}}\ldots,$$

$$\text{if } Q_1 = Q_{1\mathrm{e}} \text{ and } Q_2 = Q_{2\mathrm{e}} \;\Rightarrow\; P_1\left[\overleftarrow{P_2}\right]^{\infty} = \mathrm{C}Q_{1\mathrm{e}}\mathrm{L}Q_{2\mathrm{e}}\mathrm{R}Q_{2\mathrm{e}}\mathrm{R}Q_{2\mathrm{e}}\mathrm{R}Q_{2\mathrm{e}}\ldots,$$

and indeed in all the cases the pattern $P_1\left[\overleftarrow{P_2}\right]^{\infty}$ has a preperiod and it is eventually periodic. Note that the periodic part of the resulting pattern has even L parity because $\mathrm{L}Q_{2\mathrm{o}}$ and $\mathrm{R}Q_{2\mathrm{e}}$ have even L parity. Also note that the period of this periodic part is equal to the period of P_2. In the same way, if we add indefinitely the pattern P_2 to P_1 always toward the right, we obtain a pattern with periodic part $\mathrm{L}Q_{2\mathrm{o}}$ or $\mathrm{R}Q_{2\mathrm{e}}$ and even L parity. Although the form of the pattern $P_1\left[\overleftarrow{P_2}\right]^{\infty}$ or $P_1\left[\vec{P_2}\right]^{\infty}$ is the same as that of a Misiurewicz point, the pattern can correspond to a Misiurewicz point or not. We can determine a posteriori when the resulting pattern corresponds to a Misiurewicz point. If the inverse path of the resulting pattern is a l.i.p. (legal inverse path) [4] the pattern corresponds to a Misiurewicz point [15]; but if the inverse path of the resulting pattern is not a l.i.p. the pattern does not correspond to a Misiurewicz point.

Let us see three simple examples. First, let $P_1 = \mathrm{CLRL}^4$ and $P_2 = \mathrm{CLR}^2$ be the patterns of two superstable periodic orbits in the family of leftward maps of Eq. (2), corresponding to the parameters values $c_1 = -1.574889\ldots$ and $c_2 = -1.940799\ldots$. If we add indefinitely the pattern P_2 to P_1 always toward the left we have $P_1\left[\overleftarrow{P}_2\right]^{\infty} = (\mathrm{CLRL}^4\mathrm{R})\mathrm{LR}^2\mathrm{L}$, a supposed $M_{8,4}$ Misiurewicz point. If we search in the table of Misiurewicz points of Ref. [15] we find fifty six $M_{8,4}$, but no one has the symbolic sequence $(\mathrm{CLRL}^4\mathrm{R})\mathrm{LR}^2\mathrm{L}$. Therefore this pattern is not the pattern of a Misiurewicz point. Second, if we take the same pattern $P_1 = \mathrm{CLRL}^4$ but the pattern P_2 is now $P_2 = \mathrm{CLRL}$, corresponding to the parameter value $c_2 = -1.310702\ldots$, we have $P_1\left[\overleftarrow{P}_2\right]^{\infty} = (\mathrm{CLRL}^3)\mathrm{LR}$, which as can be seen in the table of Ref. [15] is the Misiurewicz point $M_{6,2}^{(1)}$ located at the parameter value $c = -1.589484\ldots$. Third, if $P_1 = \mathrm{CLRL}^4$ and $P_2 = \mathrm{CLRL}$, and we add indefinitely the pattern P_2 to P_1 always toward the right we have $P_1\left[\vec{P}_2\right]^{\infty} = (\mathrm{CLRL}^5)\mathrm{LR}$ that corresponds to the Misiurewicz point $M_{8,2}^{(1)}$ located at the parameter value $c = -1.560984\ldots$ [15].

Therefore $P_1\left[\vec{P}_2\right]^{\infty}$ (where " $\leftrightarrow$" means toward the left or toward the right) is a pattern with a preperiod and a period, but sometimes is a Misiurewicz point and sometimes not. To determine a priori in which case we are, let us see the following experimental property:

Property 2. Let *P* be a superstable periodic orbit pattern in a family of 1D quadratic maps, and let *a* be a heredity transmitter of this pattern. If we add indefinitely *a* to *P* always with the same direction the pattern obtained corresponds to a Misiurewicz point.

We have an example of this property in Fig. 1. In the upper part we can see the right hand part of the non-binary tree of descendants [25] of the superstable periodic orbits patterns $P_5 = \mathrm{CLR}^3$ and $P_6 = \mathrm{CLR}^4$ in the family of 1D quadratic maps of Eq. (2). In the lower part we can see the corresponding zone of the Mandelbrot set antenna drawn by the escape line method. As it is easy to obtain, the heredity transmitters of P_5 are $a_1 = \mathrm{CL}$, $a_2 = \mathrm{CLR}$, and $a_3 = \mathrm{CLR}^2$ [25]. If we add successively the first heredity transmitter a_1 toward the right to pattern P_5, we obtain the patterns $P_{7r} = \mathrm{CLR}^3\mathrm{L}^2$, $P_{9r} = \mathrm{CLR}^3\mathrm{L}^4$, $P_{11r} = \mathrm{CLR}^3\mathrm{L}^6$, $\ldots$. In the limit, the pattern $(\mathrm{CLR}^3)\mathrm{L}$ corresponding to the Misiurewicz point $M_{5,1}^{(1)}$ is obtained. Now, if we add successively the heredity transmitter a_1 toward the left, the pattern $(\mathrm{CLR}^4)\mathrm{L}$ corresponding to the Misiurewicz point $M_{6,1}^{(3)}$ is obtained as the limit of the patterns

$P_{7l} = \mathrm{CLR^4L}$, $P_{9l} = \mathrm{CLR^4L^3}$, … . Both patterns, $(\mathrm{CLR^3})\mathrm{L}$ and $(\mathrm{CLR^4})\mathrm{L}$, correspond to characteristic Misiurewicz points [14] because they have the same period that the chaotic band where they are. On the other hand, the pattern P_5 belongs to the chaotic band $\mathbf{B}_0$ whose gene is the pattern C [23]. The first harmonic of this gene is $\mathrm{C} \mp \mathrm{C} = \mathrm{CL}$, i.e. the first heredity transmitter a_1. Again, we can enunciate a new experimental property:

Property 3. Let *P* be a superstable periodic orbit pattern in a family of 1D quadratic maps, and let a_1 be the pattern of the first heredity transmitter of *P*. The result of adding indefinitely a_1 to *P* always with the same direction, on the right or on the left, is the pattern of a characteristic Misiurewicz point.

Let $\mathbf{G}_0 = \mathrm{C}$, $\mathbf{G}_1 = \mathrm{CL}$, $\mathbf{G}_2 = \mathrm{CLRL}$,… be the genes of the chaotic bands $\mathbf{B}_0$, $\mathbf{B}_1$, $\mathbf{B}_2$,… in a family of 1D quadratic maps [23]. The first heredity transmitter of a superstable periodic orbit pattern located at the chaotic band $\mathbf{B}_i$ is $H^{(1)}(\mathbf{G}_i)$, i. e. $H^{(1)}(\mathbf{G}_0) = \mathrm{CL}$, $H^{(1)}(\mathbf{G}_1) = \mathrm{CLRL}$, $H^{(1)}(\mathbf{G}_2) = \mathrm{CLRL^3RL}$, …. Thus, we can obtain the two characteristic Misiurewicz points patterns (on the left and on the right) of a given pattern *P* located at a given chaotic band. So, the characteristic Misiurewicz points patterns corresponding to the pattern $\mathrm{CLRL^5}$ located at the chaotic band $\mathbf{B}_1$ are $\mathrm{CLRL^5}\left[\overleftarrow{\mathrm{C}}\mathrm{LRL}\right]^{\infty} = (\mathrm{CLRL^6})\mathrm{LR}$, corresponding to a $M_{9,2}$ Misiurewicz point, and $\mathrm{CLRL^5}\left[\vec{\mathrm{C}}\mathrm{LRL}\right]^{\infty} = (\mathrm{CLRL^4})\mathrm{LR}$, corresponding to the Misiurewicz point $M_{7,2}^{(1)}$ located at the parameter value $c = -1.509171\ldots$ [15].

As can clearly be seen in Fig. 1 all the descendants of $\mathrm{CLR^3}$, which constitute the tree of pattern $\mathrm{CLR^3}$, are between the characteristic Misiurewicz points whose patterns are $(\mathrm{CLR^4})\mathrm{L}$ and $(\mathrm{CLR^3})\mathrm{L}$, and there are no other superstable periodic orbit patterns inside this interval but the descendants of $\mathrm{CLR^3}$. In a family of 1D quadratic maps every superstable periodic orbit pattern *P* has two characteristic Misiurewicz points associated, one on the left and one on the right, which have the property that between them all the descendants of *P*, and only these, are present.

We can justify property 3. Let $\mathbf{G} = \mathrm{CQ}$ be the gene of the chaotic band where the pattern P is located. The pattern a_1 has to be the form $a_1 = \mathrm{C}Q\mathrm{X}Q$ (X is the letter R or L) because it is the first harmonic of $\mathbf{G}$. Let us call Q_e a Q whose L parity is even, and Q_o a Q whose L parity is odd. We have two cases. By substituting for each one of them, we obtain

$$\text{if } Q = Q_\text{o} \;\Rightarrow\; [\overleftarrow{a}_1]^\infty = [\overleftarrow{\text{C}}\text{Q}_\text{o}\text{R}Q_\text{o}]^\infty = \overleftarrow{\text{C}}\text{Q}_\text{o}\text{R}Q_\text{o}\ \text{L}Q_\text{o}\text{R}Q_\text{o}\ \text{R}Q_\text{o}\text{R}Q_\text{o}\ \text{R}Q_\text{o}\text{R}Q_\text{o}\ldots,$$

$$[\vec{a}_1]^\infty = [\vec{\text{C}}\text{Q}_\text{o}\text{R}Q_\text{o}]^\infty = \vec{\text{C}}\text{Q}_\text{o}\text{R}Q_\text{o}\ \text{R}Q_\text{o}\text{R}Q_\text{o}\ \text{R}Q_\text{o}\text{R}Q_\text{o}\ \text{R}Q_\text{o}\text{R}Q_\text{o}\ldots,$$

$$\text{if } Q = Q_\text{e} \;\Rightarrow\; [\overleftarrow{a}_1]^\infty = [\overleftarrow{\text{C}}\text{Q}_\text{e}\text{L}Q_\text{e}]^\infty = \overleftarrow{\text{C}}\text{Q}_\text{e}\text{L}Q_\text{e}\ \text{R}Q_\text{e}\text{L}Q_\text{e}\ \text{L}Q_\text{e}\text{L}Q_\text{e}\ \text{L}Q_\text{e}\text{L}Q_\text{e}\ldots,$$

$$[\vec{a}_1]^\infty = [\vec{\text{C}}\text{Q}_\text{e}\text{L}Q_\text{e}]^\infty = \vec{\text{C}}\text{Q}_\text{e}\text{L}Q_\text{e}\ \text{L}Q_\text{e}\text{L}Q_\text{e}\ \text{L}Q_\text{e}\text{L}Q_\text{e}\ \text{L}Q_\text{e}\text{L}Q_\text{e}\ldots,$$

and the resulting Misiurewicz point patterns $P[\overleftarrow{a}_1]^\infty$, $P[\vec{a}_1]^\infty$ and the gene **G** have the same period. Therefore $P[\overleftarrow{a}_1]^\infty$ and $P[\vec{a}_1]^\infty$ are patterns of characteristic Misiurewicz points. Moreover, in a family of leftward maps the L parity of the periodic part of the patterns of the two characteristic Misiurewicz points corresponding to a given superstable periodic orbit pattern *P* is odd, and the period of this periodic part is the half as great as the period of the first heredity transmitter of the pattern *P*.

Now, let us consider the pattern CLR^4 in Fig. 1. This pattern has four heredity transmitters, CL, CLR, CLR^2 and CLR^3, where CL is the first harmonic of the gene of the chaotic band $\mathbf{B}_0$ where *P* is. The pattern of the right hand characteristic Misiurewicz point corresponding to pattern CLR^4 is $\text{CLR}^4[\vec{\text{C}}\text{L}]^\infty = (\text{CLR}^4)\text{L}$. But we have seen above that the pattern of the left hand characteristic Misiurewicz point corresponding to pattern CLR^3 is $\text{CLR}^3[\overleftarrow{\text{C}}\text{L}]^\infty = (\text{CLR}^4)\text{L}$. Then, the Misiurewicz characteristic point pattern $(\text{CLR}^4)\text{L}$ separates the tree generated by the pattern CLR^3 from the tree generated by the pattern CLR^4. Let us see now in Fig. 1 the composition of the period-5 superstable orbit pattern $P_5 = \text{CLR}^3$ with its second heredity transmitter $a_2 = \text{CLR}$. If we add successively a_2 to P_5 on the right, we obtain the patterns of superstable periodic orbits $P_{8r} = \text{CLR}^3\text{L}^2\text{R}$, $P_{11r} = \text{CLR}^3\overline{\text{L}^2\text{R}}^2, \ldots$ In the limit, the Misiurewicz point pattern $(\text{CLR}^2)\text{RL}^2$, corresponding to $M_{3,4}^{(3)}$ and located at the parameter value $c = -1.983681\ldots$ [15], is obtained. If now we add successively a_2 to P_5 on the left, we obtain the Misiurewicz point pattern $(\text{CLR}^4)\text{LRL}$, corresponding to $M_{6,3}^{(8)}$ located at $c = -1.989253\ldots$ [15], which is the limit of the patterns of superstable periodic orbits $P_{8l} = \text{CLR}^4\text{LR}$, $P_{11l} = \text{CLR}^4\text{LRL}^2\text{R}, \ldots$ The patterns $(\text{CLR}^2)\text{RL}^2$ and $(\text{CLR}^4)\text{LRL}$ correspond to non-characteristic Misiurewicz points [14] because their periods (three) are different from the period of the chaotic band where they are located (which is one). Note that the period of the periodic part of the resulting Misiurewicz point pattern is the same as the period of the second heredity transmitter CLR. In Fig. 1 other two non-characteristic Misiurewicz points are obtained by adding the third heredity transmitter $a_3 = \mathrm{CLR}^2$ to P_5. According to this, we can give the following property:

Property 4. Let *P* be a superstable periodic orbit pattern in a family of 1D quadratic maps and let a_i $(i \neq 1)$ be the pattern of one of its heredity transmitters except for the first. The result of adding indefinitely a_i to *P* always with the same direction, on the right or on the left, is the pattern of a non-characteristic Misiurewicz point.

Let us see now the third method, the generation of a Misiurewicz point from the composition with a short cut.

4. Composition of a pattern with a short cut

Let $P_1 = \mathrm{CLR}^4$ be the pattern of a period-6 superstable orbit in the family of quadratic maps of Eq. (2). If we add indefinitely the pattern $P_2 = \mathrm{CLRLR}$ to P_1 on the right it is possible that the pattern of a Misiurewicz point is not obtained because P_2 is not a heredity transmitter of P_1 nor even corresponds to a superstable periodic orbit pattern (the inverse path of P_2 is not a l.i.p.). But in this case we obtain $\mathrm{CLR}^4[\vec{\mathrm{C}}\mathrm{LRLR}]^\infty =$ $(\mathrm{CLR}^4\mathrm{L})\mathrm{LRLR}^2$, the Misiurewicz point $\mathrm{M}_{7,5}^{(54)}$ located at the parameter value $-1.994548\ldots$ [15]. Therefore, it is clear that a Misiurewicz point can also be obtained from a pattern by adding indefinitely, on the right or on the left, other pattern which is neither itself nor one of its heredity transmitters.

Let us consider the pattern *P* of a superstable periodic orbit and the tree of its descendants in a family of 1D quadratic maps. We shall call “short cut” a direct way from the pattern to a descendant without going through heredity transmitters. We have verified that the following property always takes place:

Property 5. Let *P* be a superstable periodic orbit pattern in a family of 1D quadratic maps and let *S* be the pattern of a short cut in the tree of descendents of *P*. If we add indefinitely the pattern *S* to pattern *P*, always toward a same direction, the resulting pattern corresponds to a Misiurewicz point.

In the upper part of Fig. 2, we can see the right hand part of the non-binary tree of descendants of the superstable periodic orbit pattern $P_6 = \mathrm{CLR}^4$ in the family of 1D quadratic leftward maps of Eq. (2). Two short cuts are also shown in thin lines. In the lower part of Fig. 2 we can see the corresponding zone of the Mandelbrot set antenna drawn by the escape line method. The pattern of a short cut can be calculated by subtracting the superstable periodic orbit pattern of the short cut origin from the superstable periodic orbit pattern of the short cut destination, and changing the first letter by a C with an arrow in the proper direction. So, the pattern of the short cut from $P_6 = \mathrm{CLR}^4$ to $P_{10} = \mathrm{CLR}^4\mathrm{L}^2\mathrm{RL}$ is $\mathrm{CLR}^4\mathrm{L}^2\mathrm{RL} - \mathrm{CLR}^4 = \vec{\mathrm{C}}\mathrm{LRL}$. The pattern of a short cut can be also expressed in function of heredity transmitters. So, the short cut from P_6 to P_{10} also is $\vec{\mathrm{C}}\mathrm{L}\overleftarrow{\mathrm{C}}\mathrm{L}$ (see Fig. 2). Whether the L parity of P is odd or even it is easy verify that $P\vec{\mathrm{C}}\mathrm{L}\overleftarrow{\mathrm{C}}\mathrm{L} = P\vec{\mathrm{C}}\mathrm{LRL}$, and therefore the short cuts $\vec{\mathrm{C}}\mathrm{L}\overleftarrow{\mathrm{C}}\mathrm{L}$ and $\vec{\mathrm{C}}\mathrm{LRL}$ are the same.

In Fig. 2 we show two graphic examples of Misiurewicz points generation via short cuts. In the first one the short cut $\vec{\mathrm{C}}\mathrm{L}\overleftarrow{\mathrm{C}}\mathrm{L}$ is infinitely added to the superstable periodic orbit pattern CLR^4 to obtain the pattern $(\mathrm{CLR}^4\mathrm{L})\mathrm{LR}$ corresponding to the Misiurewicz point $\mathrm{M}_{7,2}^{(8)}$ located at $c = -1.994779\ldots$ [15]. In the second example, the short cut $\vec{\mathrm{C}}\mathrm{LR}\overleftarrow{\mathrm{C}}\mathrm{L}$ is infinitely added to pattern CLR^4 to obtain the Misiurewicz point pattern $(\mathrm{CLR}^4\mathrm{L})\mathrm{LR}^2\mathrm{LR}$ corresponding to $\mathrm{M}_{7,5}^{(56)}$ located at $c = -1.995649\ldots$ [15]. We have seen that the period of the periodic part of the resulting Misiurewicz point pattern is the same as the period of the short cut if this has odd L parity, and is the half as greater as the period of the short cut if this has even L parity.

We also can infinitely add the composition pattern $\vec{\mathrm{C}}\mathrm{L}\vec{\mathrm{C}}\mathrm{L}$ to CLR^4. But $\vec{\mathrm{C}}\mathrm{L}\vec{\mathrm{C}}\mathrm{L}$ is not a short cut but a multiple of the heredity transmitter $\vec{\mathrm{C}}\mathrm{L}$. In a short cut the heredity transmitters are different or they have different directions. It is clear that we obtain a same Misiurewicz point pattern if we add indefinitely to a superstable periodic orbit pattern one of its heredity transmitters or a multiple of this one. In this way we obtain $(\mathrm{CLR}^4)\mathrm{L}$ corresponding to the Misiurewicz point $\mathrm{M}_{6,1}^{(3)}$ located at $c = -1.993545\ldots$ [15]. On the other hand we also obtain a same Misiurewicz point pattern if we add indefinitely to a superstable periodic orbit pattern either a short cut or a multiple of this one.

The infinite composition of a superstable periodic orbit pattern with itself, with an heredity transmitter, or with a short cut, are the only three ways we have found to obtain a Misiurewicz point pattern. Later we shall see the inverse process, that is, when the pattern of a Misiurewicz point is given we shall find the generating pattern and the pattern indefinitely added. But let us note some specifications on short cuts and other related issues before beginning this inverse process study.

5. Some specifications

5.1 *Short cuts study*

Let us consider the family of quadratic maps of Eq. (2). Let $P = \mathrm{CLR}^4$ be a superstable periodic orbit pattern and let $S = \overleftarrow{\mathrm{C}}\mathrm{L}\overrightarrow{\mathrm{C}}\mathrm{LR}\overrightarrow{\mathrm{C}}\mathrm{L}$ be a short cut which is made up of heredity transmitters of *P*. We have $PS = \mathrm{CLR}^4\underline{\mathrm{R}}\mathrm{L}\underline{\mathrm{R}}\mathrm{LR}\underline{\mathrm{L}}\mathrm{L}$, where the letters that come from a C are underlined. When we go from the first heredity transmitter to the second one in the short cut, there is a change of direction (from $\leftarrow$ to $\rightarrow$), there is not a change of letter ($\underline{\mathrm{R}}$ to $\underline{\mathrm{R}}$), and the number of L's of the initial heredity transmitter ($\underline{\mathrm{R}}\mathrm{L}$) is odd. If we assign an "1" to a change of direction, a change of letter and an odd number of L's, and we assign a "0" to a non change of direction, a non change of letter and an even number of L's, we have the sum $1+0+1$ that is even. When we go from the second heredity transmitter to the third one in the short cut, there is not a change of direction (from $\rightarrow$ to $\rightarrow$), there is a change of letter ($\underline{\mathrm{R}}$ to $\underline{\mathrm{L}}$), and the number of L's of the initial heredity transmitter ($\underline{\mathrm{R}}\mathrm{LR}$) is odd: the sum $0+1+1$ is also even. It is easy to check that the following property always takes place:

Property 6. Let *P* be a superstable periodic orbit pattern in a family of 1D quadratic maps. Let *S* be a short cut which is made up of heredity transmitters of *P*. Let us consider the directions of the heredity transmitters and the letters that come from the C's in the composition pattern *PS*. When we go from a heredity transmitter to the next one, in the short cut, if we assign a "1" to a change of direction, a change of letter and an odd number of L's in the initial heredity transmitter, and we assign a "0" to a non change of direction, a non change of letter and an even number of L's, the sum of the direction change, letter change and number of L's is even.

It is easy to deduce from this property if a change of direction occurs inside a short cut, because if the sum of the letter change and the number of L's is even, no change of direction take place so that it goes on with even; and if this sum is odd, a change of direction takes place because odd plus odd is even. Let us consider the above composition $PS = \mathrm{CLR}^4\underline{\mathrm{R}}\mathrm{L}\underline{\mathrm{R}}\mathrm{LR}\underline{\mathrm{L}}\mathrm{L}$. We agree that the direction of the first heredity transmitter of the short cut is *d* (we call $\overline{d}$ to the opposite direction). When we go from the first to the second heredity transmitter there is no change of letter and the number of L's in the first one is odd; therefore there is a change of direction, and the second heredity transmitter has the direction $\overline{d}$. When we go from the second to the third heredity transmitter, there is a change of letter and the number of L's in the second heredity transmitter is odd; therefore there is no change of direction and the third heredity transmitter has the direction $\overline{d}$.

To simplify, sometimes we do not write the complete pattern of a short cut but either what we call its "basic composition", "direction basic composition" or "vectorial composition”. These simplified compositions only show the periods of the heredity transmitters that compose the short cut instead of its patterns. In the $S = \underline{\mathrm{R}}\mathrm{L}\underline{\mathrm{R}}\mathrm{LR}\underline{\mathrm{L}}\mathrm{L}$ case we have the basic composition $S_b = 232$, the direction basic composition $S_d = 2_d 3_{\bar{d}} 2_{\bar{d}}$, and the vectorial compositions $S_{\vec{v}} = \overleftarrow{2}\vec{3}\vec{2}$ and $S_{\overleftarrow{v}} = \vec{2}\overleftarrow{3}\overleftarrow{2}$.

5.2. *Preperiod simplification.*

As is known, if a Misiurewicz point is *n*-preperiodic and eventually *p*-periodic, it is also $(n+q)$-preperiodic and eventually $(p \cdot r)$-periodic, where $q = 0,1,2,\ldots$ and $r = 1,2,3,\ldots$ [14]. All the infinite ways of representing a Misiurewicz point form an equivalence class. When the pattern of a Misiurewicz point is given, it is always referred to the representative of the equivalence class, which is obtained when $q = 0$ and $r = 1$. Hence, the representative of the class is the one which has the simpler form, that is, the lesser preperiod and period. Having taken this into account, if the pattern of a Misiurewicz point is not in its simpler form, we have to simplify it.

For that, it is useful to bear in mind the following: when a Misiurewicz point pattern is written in its simpler form, the preperiod and the period cannot finish with the same letter. Indeed, if both the preperiod and the period end in a L (it would be the same if both end in a R) the Misiurewicz point $M_{n,p}$ has the pattern $(\mathrm{CX}_1 \ldots \mathrm{X}_{n-2}\mathrm{L})\mathrm{Y}_1 \ldots \mathrm{Y}_{p-1}\mathrm{L}$, where X is a L or R in the preperiod and Y is a L or R in the period. In this case, if we expand and regroup, we can write this Misiurewicz point pattern as $(\mathrm{CX}_1 \ldots \mathrm{X}_{n-2})\mathrm{LY}_1 \ldots \mathrm{Y}_{p-1}$ that corresponds to a Misiurewicz point $M_{n-1,p}$. If X_{n-2} and Y_{p-1} are the same letter we would simplify again. In each simplification the length of the period remain the same and the length of the preperiod decreases one unit.

5.3. *Related short cuts*

Let *P* be a superstable periodic orbit pattern and let $S = a_1 a_2 \ldots a_k$ be a short cut constituted by *k* heredity transmitters of the pattern *P* in a family of 1D quadratic maps. As it is easy to see, we obtain the same Misiurewicz point pattern if we compose the pattern *P* with the short cut $a_1 a_2 \ldots a_k$, the pattern Pa_1 with the short cut $a_2 a_3 \ldots a_k a_1$, the pattern $Pa_1 a_2$ with the short cut $a_3 a_4 \ldots a_k a_1 a_2$, and so on. The short cuts $a_1 a_2 \ldots a_k$, $a_2 a_3 \ldots a_k a_1$, $a_3 a_4 \ldots a_k a_1 a_2$, $\ldots$ are related short cuts.

After these considerations, we shall analyze the inverse process.

6. Inverse process

As we saw in § 2, 3 and 4 there are three ways to obtain a Misiurewicz point pattern starting from an augend (the superstable periodic orbit pattern) and an addend (the pattern itself, one of its heredity transmitters, or a short cut) which is repeatedly added to the augend. Given a Misiurewicz point pattern, the inverse process consists in finding the augend and the addend from which such Misiurewicz point pattern is obtained. We show here the four steps in which we have divided the procedure and that we have to follow in order to reach this purpose. We will explain these steps for the case of a family of 1D quadratic leftward maps; the extension to a family of 1D quadratic rightward maps is immediate.

6.1 *First step*: *finding the period of the addend*

In §2 we have seen that the periodic part of a Misiurewicz point pattern obtained by the composition of a pattern with itself has even L parity and has a period equals to the period of the pattern. In §3 we have seen that the periodic part of a Misiurewicz point pattern obtained by the composition of a superstable periodic orbit pattern with one of its heredity transmitters except for the first one has even L parity and has a period equal to the period of the heredity transmitter. In §3 we have also seen that the periodic part of a Misiurewicz point pattern obtained by the composition of a superstable periodic orbit pattern with the first of its heredity transmitters has odd L parity and has a period that is the half as great as the period of the first heredity transmitter. Finally, in §4 we have seen that the periodic part of a Misiurewicz point pattern obtained by the composition of a superstable periodic orbit pattern with a short cut is the same as the period of the short cut if the periodic part of the Misiurewicz point pattern has even L parity, and is the half as greater as the period of the short cut if the periodic part of Misiurewicz point pattern has odd L parity. Hence, we have the following property:

Property 7. Let us consider a family of 1D quadratic leftward maps. If the L parity of the periodic part of a Misiurewicz point pattern is even, the period of the addend is equal to the period of the periodic part. If the L parity of the periodic part of the Misiurewicz point pattern is odd, the period of the addend is twice the period of the periodic part.

6.2 *Second step: finding low period heredity transmitters of superstable periodic orbit patterns located in the neighborhood of the Misiurewicz point*

Low period heredity transmitters refers to heredity transmitters whose periods are less or equal to the period of the addend (the period found in §6.1). Since a Misiurewicz point pattern is the limit of a series of superstable periodic orbit patterns, we can approach the Misiurewicz point pattern to one of these superstable periodic orbits patterns. As it is easy to see, in the neighborhood of the Misiurewicz point all the superstable periodic orbit patterns have the same low period heredity transmitters.

6.3 *Third step: finding the pattern of the addend*

First, we begin by seeing if the addend is the augend itself. In this case, starting from property 7, the L parity of the periodic part of the Misiurewicz point pattern must be even and the addend pattern is the low period heredity transmitter whose period coincides with the period of the periodic part. If that occurs, we have finished this step. Second, we must see if the addend is a low period heredity transmitter. In this case, the addend pattern is the low period heredity transmitter whose period coincides with the period given by property 7, and we have finished this step. Third, if the addend pattern is a short cut, we have to find both the basic composition and the direction basic composition before finding the short cut pattern. In order to find the basic composition of the short cut, we have to compose low period heredity transmitters until the sum of its periods equals to the one found in §6.1. Now, by taking into account property 6 of §5.1, we can easily find the direction basic composition and, from it, the short cut pattern.

6.4 *Fourth step: finding the pattern of the augend*

To accomplish this goal, we write in a line the Misiurewicz point pattern in a developed form (i.e. without brackets and with the periodic part repeated some times) and in a line just under the first one we write the addend as on the left as possible in such a manner that its letters (L's and R's) and the letters on the upper line coincide and when we apply the leftward rule to $\overleftarrow{C}$ (or the rightward rule to $\vec{C}$), this $\overleftarrow{C}$ (or $\vec{C}$) turn into the letter just over it. On the left of the addend there are some letters without any correspondence: these letters are the augend.

Next, we shall get used to some examples in order to clarify the procedure and to analyze some singularities.

7. Inverse process examples

Example 1

This example treats a case where the L parity of the periodic part of the Misiurewicz point pattern is odd. Given the Misiurewicz point pattern $(CLR)RL$, in a family of 1D quadratic leftward maps, let us look for the augend and addend patterns.

According to the first step, the period of the addend is $2 \times 2 = 4$. In accordance with the second step we have to choose a superstable periodic orbit pattern next enough to the given Misiurewicz point, for instance $\mathrm{CLRRLRLRL}$. The heredity transmitters of this pattern are CL, CLR and $\mathrm{CLR^2LRL}$. According to second step, the low period heredity transmitters are CL and CLR (because the period of $\mathrm{CLR^2LRL}$ is bigger than 4). Following the third step, since there are no heredity transmitter of period 4, the addend have to be a short cut. The only possibility is $4 = 2 + 2$, that is, a short cut composed by two heredity transmitters CL and therefore with the basic composition 22. Taking into account the periodic part of the Misiurewicz point pattern, the short cut has to be $\underline{\mathrm{R}}\mathrm{L}\underline{\mathrm{R}}\mathrm{L}$ and its direction basic composition $2_d 2_{\bar{d}}$. Hence, the two short cuts $\overleftarrow{\mathrm{C}}\mathrm{L}\overrightarrow{\mathrm{C}}\mathrm{L} = \overleftarrow{\mathrm{C}}\mathrm{LRL}$ and $\overrightarrow{\mathrm{C}}\mathrm{L}\overleftarrow{\mathrm{C}}\mathrm{L} = \overrightarrow{\mathrm{C}}\mathrm{LRL}$ are possible. According to the fourth step we write

C L R R L R L R L …
· · · $\overleftarrow{\mathrm{C}}$ L R L …

that gives the augend CLR and the addend $\overleftarrow{\mathrm{C}}\mathrm{LRL}$, and

C L R R L R L R L …
· · · · · $\overrightarrow{\mathrm{C}}$ L R L …

that gives the augend $\mathrm{CLR^2L}$ and the addend $\overrightarrow{\mathrm{C}}\mathrm{LRL}$.

Hence, we reach the Misiurewicz point pattern (CLR)RL either by starting from the superstable periodic orbit pattern CLR and by adding indefinitely the short cut $\overleftarrow{\mathrm{C}}\mathrm{LRL} = \overleftarrow{\mathrm{C}}\mathrm{L}\overrightarrow{\mathrm{C}}\mathrm{L}$ or by starting from the superstable periodic orbit pattern $\mathrm{CLR^2L}$ and by adding indefinitely the short cut $\overrightarrow{\mathrm{C}}\mathrm{LRL} = \overrightarrow{\mathrm{C}}\mathrm{L}\overleftarrow{\mathrm{C}}\mathrm{L}$, as can be seen in Fig. 3.

Example 2

This example treats a case where the L parity of the periodic part of the Misiurewicz point pattern is even. Let us search the augend and the addend that originates the Misiurewicz point pattern $(\mathrm{CLR}^2)\mathrm{L}^3\mathrm{RL}$ in a family of 1D quadratic leftward maps. First step: as the periodic part of this pattern has even L parity, the period of the addend must to be 5. Second step: the low period heredity transmitters in the neighborhood of the Misiurewicz point are CL, CLR and $\mathrm{CLR}^2\mathrm{L}$. The heredity transmitter $\mathrm{CLR}^2\mathrm{L}$ is not valid because it has two R's and the periodic part of the Misiurewicz point pattern has only one; therefore, the addend has to be a short cut. As $5 = 2 + 3$, the possible basic compositions are $S_{b1} = 23$ and $S_{b2} = 32$. Let us consider S_{b1}. Third step: as the total number of L's in the periodic part of Misiurewicz point pattern is 4, the short cut CLCLR has to be $\underline{\mathrm{L}}\mathrm{L}\underline{\mathrm{L}}\mathrm{LR}$ with direction basic composition $S_{d1} = 2_d 3_d$. Therefore, the vectorial compositions of the short cut are $S_{\overleftarrow{v}1} = \overleftarrow{\mathrm{C}}\mathrm{L}\overleftarrow{\mathrm{C}}\mathrm{LR} = \overleftarrow{\mathrm{C}}\mathrm{L}^3\mathrm{R}$ and $S_{\vec{v}1} = \overrightarrow{\mathrm{C}}\mathrm{L}\overrightarrow{\mathrm{C}}\mathrm{LR} = \overrightarrow{\mathrm{C}}\mathrm{L}^3\mathrm{R}$. Four step: we write

$$\begin{array}{l} \mathrm{C\ L\ R\ R\ L^3\ R\ L\ L^3\ R\ L} \ldots \\ \cdot\ \ \cdot\ \ \cdot\ \overleftarrow{\mathrm{C}}\ \mathrm{L}^3\ \mathrm{R} \ldots \end{array} ,$$

that gives the augend CLR. No addend adjusts to $S_{\vec{v}1} = \overrightarrow{\mathrm{C}}\mathrm{L}^3\mathrm{R}$.

We reach the Misiurewicz point pattern $(\mathrm{CLR}^2)\mathrm{L}^3\mathrm{RL}$ by starting from superstable periodic orbit pattern CLR and by adding indefinitely the short cut $\overleftarrow{\mathrm{C}}\mathrm{L}\overleftarrow{\mathrm{C}}\mathrm{LR}$, as we can see in Fig. 3. We shall not treat, as we said in §7.2 , the case $S_{b2} = 32$. But, since it is a related short cut, we can predict the result. The superstable periodic orbit pattern must to be the composition pattern $\mathrm{CLR}\overleftarrow{\mathrm{C}}\mathrm{L} = \mathrm{CLR}^2\mathrm{L}$, and the short cut must to be $\overleftarrow{\mathrm{C}}\mathrm{LR}\overleftarrow{\mathrm{C}}\mathrm{L} = \overleftarrow{\mathrm{C}}\mathrm{LRL}^2$.

Example 3

This example treats related short cuts. Let $(\mathrm{CLRL}^5)\mathrm{L}^3\mathrm{R}$ be a Misiurewicz point pattern in a family of 1D quadratic leftward maps, and let us look for the augend and addend patterns. First step: as the L parity of periodic part is odd, the period of the addend is $4 \times 2 = 8$. Second step: we must choose a superstable periodic orbit pattern close enough to the Misiurewicz point pattern. Candidate patterns are $\mathrm{CLRL}^8\mathrm{R}$, $\mathrm{CLRL}^8\mathrm{RL}$, $\mathrm{CLRL}^8\mathrm{RL}^2$, …. We chose $\mathrm{CLRL}^8\mathrm{RL}$ (its inverse path is a l.i.p.). The heredity transmitters of this pattern are CL and CLRL^8, but only CL is a low period heredity transmitter. Third step: the addend has to be a short cut with basic composition 2222. According to the periodic part of the Misiurewicz point, the short cut has to have two R's and six L's. The only possible short cuts are $\underline{R}L\underline{L}L\underline{R}L\underline{L}L$ and $\underline{L}L\underline{R}L\underline{L}L\underline{R}L$ which are related short cuts.

From the first short cut $\underline{R}L\underline{L}L\underline{R}L\underline{L}L$ we obtain the direction basic composition $S_{d1} = 2_d 2_d 2_{\bar{d}} 2_{\bar{d}}$ that gives the vectorial composition on the left $S_{\overleftarrow{v}1} = \overleftarrow{C}L\overleftarrow{C}L\vec{C}L\vec{C}L = \overleftarrow{C}L^3RL^3$ and the vectorial composition on the right $S_{\vec{v}1} = \vec{C}L\vec{C}L\overleftarrow{C}L\overleftarrow{C}L = \vec{C}L^3RL^3$. According to the fourth step, we write

$$\begin{array}{l} C\ L\ R\ L^5\ L^3\ R\ L^3\ R\ L^3\ R\ \ldots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \overleftarrow{C}\ L^3\ R\ L^3\ \ldots \end{array}$$

that gives the augend $CLRL^8$ and the addend $\overleftarrow{C}L^3RL^3$, and

$$\begin{array}{l} C\ L\ R\ L^4\ L\ L^3\ R\ L^3\ R\ \ldots \\ \cdot\ \cdot\ \cdot\ \cdot\ \vec{C}\ L^3\ R\ L^3\ \ldots \end{array}$$

that gives the augend $CLRL^4$ and the addend $\vec{C}L^3RL^3$.

We reach the Misiurewicz point pattern $(CLRL^5)L^3R$ either by starting from the superstable periodic orbit pattern $CLRL^8$ and by adding indefinitely the short cut $S_{\overleftarrow{v}1} = \overleftarrow{C}L\overleftarrow{C}L\vec{C}L\vec{C}L$ or by starting from the superstable periodic orbit pattern $CLRL^4$ and by adding indefinitely the short cut $S_{\vec{v}1} = \vec{C}L\vec{C}L\overleftarrow{C}L\overleftarrow{C}L$, as can be seen in Fig. 4.

From the second short cut $\underline{L}L\underline{R}L\underline{L}L\underline{R}L$ the direction basic composition is $S_{d2} = 2_d 2_{\bar{d}} 2_{\bar{d}} 2_d$, the vectorial composition on the left is $S_{\overleftarrow{v}2} = \overleftarrow{C}L\vec{C}L\vec{C}L\overleftarrow{C}L = \overleftarrow{C}LRL^3RL$ and the vectorial composition on the right is $S_{\vec{v}2} = \vec{C}L\overleftarrow{C}L\overleftarrow{C}L\vec{C}L = \vec{C}LRL^3RL$. We can write

$$\begin{array}{l} C\ L\ R\ L^5\ L^3\ R\ L\ L\ L\ R\ L^3\ R\ L\ L\ L\ \ldots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \overleftarrow{C}\ L\ R\ L^3\ R\ L\ \ldots \end{array}$$

that gives the augend $CLRL^8RL$ and the addend $\overleftarrow{C}LRL^3RL$, and

$$\begin{array}{l} C\ L\ R\ L^5\ L\ L\ L\ R\ L^3\ R\ L\ L\ L\ \ldots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \vec{C}\ L\ R\ L^3\ R\ L\ \ldots \end{array}$$

that gives the augend $CLRL^6$ and the addend $\vec{C}LRL^3RL$.

We reach the Misiurewicz point pattern $(CLRL^5)L^3R$ either by starting from the superstable periodic orbit pattern $CLRL^8RL$ and by adding indefinitely the short cut

$S_{\overleftarrow{v}2} = \overleftarrow{\mathrm{C}}\mathrm{L}\overrightarrow{\mathrm{C}}\mathrm{L}\overrightarrow{\mathrm{C}}\mathrm{L}\overleftarrow{\mathrm{C}}\mathrm{L}$ or by starting from the superstable periodic orbit pattern CLRL^6 and by adding indefinitely the short cut $S_{\overrightarrow{v}2} = \overrightarrow{\mathrm{C}}\mathrm{L}\overleftarrow{\mathrm{C}}\mathrm{L}\overleftarrow{\mathrm{C}}\mathrm{L}\overrightarrow{\mathrm{C}}\mathrm{L}$, as can be seen in Fig. 4.

Therefore, $\mathrm{CLRL}^4[S_{\overrightarrow{v}1}]^\infty$, $\mathrm{CLRL}^6[S_{\overrightarrow{v}2}]^\infty$, $\mathrm{CLRL}^8[S_{\overleftarrow{v}1}]^\infty$, and $\mathrm{CLRL}^8\mathrm{RL}[S_{\overleftarrow{v}2}]^\infty$ give the same Misiurewicz point pattern $(\mathrm{CLRL}^5)\mathrm{L}^3\mathrm{R}$ and $S_{\overrightarrow{v}1}$, $S_{\overrightarrow{v}2}$, $S_{\overleftarrow{v}1}$, and $S_{\overleftarrow{v}2}$ are related short cuts. From now on, we shall normally treat one short cut and we shall not look for the related short cuts which can be deduced a posteriori in a very simple manner.

Example 4

Let us see now an example to treat the only anomaly we have detected in the fourth step, the adjustment processes. Let $(\mathrm{CLRL}^6\mathrm{R})\mathrm{L}$ be a Misiurewicz point pattern. First step: as the periodic part has odd L-parity, the period of the addend is $1 \times 2 = 2$. Second step: the only low period heredity transmitter is CL. Third step: the pattern of the addend is CL. Fourth step: we write

$$\begin{array}{llllllll} \mathrm{C} & \mathrm{L} & \mathrm{R} & \mathrm{L}^6 & \mathrm{R} & \mathrm{L} & \mathrm{L} & \mathrm{L}\ldots \\ \cdot & \cdot & \cdot & \cdot & \overleftarrow{\mathrm{C}} & \mathrm{L}\ldots & & \end{array}$$

that gives the augend CLRL^6, and

$$\begin{array}{llllllllll} \mathrm{C} & \mathrm{L} & \mathrm{R} & \mathrm{L}^6 & \mathrm{R} & \mathrm{L} & \mathrm{L} & \mathrm{L} & \mathrm{L} & \mathrm{L}\ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \overrightarrow{\mathrm{C}} & \mathrm{L}\ldots & & & \end{array}$$

that gives the augend $\mathrm{CLRL}^6\mathrm{R}$.

We reach $(\mathrm{CLRL}^6\mathrm{R})\mathrm{L}$ either by starting from the superstable periodic orbit pattern CLRL^6 and by adding indefinitely the first heredity transmitter $\overleftarrow{\mathrm{C}}\mathrm{L}$, as can be seen in Fig. 4, or by starting from the pattern $\mathrm{CLRL}^6\mathrm{R}$ and by adding indefinitely the pattern $\overrightarrow{\mathrm{C}}\mathrm{L}$. Here we find an apparent anomaly: $\mathrm{CLRL}^6\mathrm{R}$ does not correspond to a superstable periodic orbit pattern because the inverse path $\mathrm{RL}^6\mathrm{RLC}$ is not a l.i.p. [4], and then it is not a true augend. The true augend is $\mathrm{CLRL}^6\mathrm{R} + \overrightarrow{\mathrm{C}}\mathrm{L} + \overrightarrow{\mathrm{C}}\mathrm{L} = \mathrm{CLRL}^6\mathrm{RL}^4$, which is the first antiharmonic of pattern CLRL^4. As is known, an antiharmonic never corresponds to a superstable periodic orbit pattern [4, 15] but it is a superstable periodic orbit patterns generator [14] as can be seen in Fig. 4. Both patterns, $\mathrm{CLRL}^6\mathrm{R}$ and $\mathrm{CLRL}^6\mathrm{RL}^4$, are not formally contradictory because we can reach the one from the other by simplifying L's. Indeed, starting from the algorithm to obtain characteristic Misiurewicz patterns [15], the right separator of the antiharmonic $\mathrm{CLRL}^6\mathrm{RL}^4$ is indeed the Misiurewicz point pattern $(\mathrm{CLRL}^6\mathrm{R})\mathrm{L}$.

Example 5

In this example a little more complex case is treated. Let us look for the augend and the addend of the Misiurewicz point pattern $(\mathrm{CLR}^5)\mathrm{LR}^4\mathrm{LR}^2\mathrm{L}^2$ in a family of 1D quadratic leftward maps. The period of the addend is 10 because the L parity of the periodic part is even. The low period heredity transmitters are: CL, CLR, CLR^2, CLR^3 and CLR^4. Since we have no period-10 heredity transmitter, the addend has to be a short cut. As $10 = 5 + 5$, then we can test the basic composition $S_{b1} = 55$. Since the periodic part of the Misiurewicz point pattern has four L's and six R's, the short cut would be $\underline{\mathrm{L}}\mathrm{LR}^3\underline{\mathrm{L}}\mathrm{LR}^3$, which is not possible because it does not coincide with the periodic part of the Misiurewicz point pattern. But also $10 = 2 + 3 + 5$, and we can now test the basic composition $S_{b2} = 235$. As we can see, the procedure is by trial and error. In this case one of the C's has to be a L and the other two ones R's. If we take R for the first one, L for the second one and R for the third one, we have $\underline{\mathrm{R}}\mathrm{L}\underline{\mathrm{L}}\mathrm{LR}\underline{\mathrm{R}}\mathrm{LR}^3$, which does not coincide with the periodic part. We take then the basic composition $S_{b3} = 325$. As with the previous case, one of the C's has to be a L and the other two ones R's. If we take now a R for the first C, a R for the second one and a L for the third one we have $\underline{\mathrm{R}}\mathrm{LR}\underline{\mathrm{R}}\mathrm{L}\underline{\mathrm{L}}\mathrm{LR}^3$, which coincides with the periodic part when it is cyclically read. From this short cut we obtain the direction basic composition $S_{d3} = 3_d 2_{\bar{d}} 5_{\bar{d}}$ and the vectorial compositions $S_{\bar{v}} = \overleftarrow{\mathrm{C}}\mathrm{LR}\overrightarrow{\mathrm{C}}\mathrm{L}\overrightarrow{\mathrm{C}}\mathrm{LR}^3$ and $S_{\bar{v}} = \overrightarrow{\mathrm{C}}\mathrm{LR}\overleftarrow{\mathrm{C}}\mathrm{L}\overleftarrow{\mathrm{C}}\mathrm{LR}^3$. We reach $(\mathrm{CLR}^5)\mathrm{LR}^4\mathrm{LR}^2\mathrm{L}^2$ by starting from $\mathrm{CLR}^5\mathrm{LR}^3$ and by adding indefinitely $\overrightarrow{\mathrm{C}}\mathrm{LR}\overleftarrow{\mathrm{C}}\mathrm{L}\overleftarrow{\mathrm{C}}\mathrm{LR}^3$ as can be seen in Fig. 5. To obtain the least period augend CLR^4, either we simplify a posteriori or we must have got right the basic composition $S_{b4} = 532$.

Example 6

To finish, let us look for the augend and the addend of a Misiurewicz point pattern which is not in the chaotic band $\mathbf{B}_0$. Let us consider the superstable periodic orbit pattern $(\mathrm{CLRL}^3\overline{\mathrm{RL}}^3\mathrm{R})\mathrm{LRL}^2$ in the chaotic band $\mathbf{B}_2$. The period of the addend is $2 \times 4 = 8$. The only low period heredity transmitter is $\mathrm{CLRL}^3\mathrm{RL}$ (the equivalent to the superstable orbit pattern CL in the chaotic band $\mathbf{B}_0$). We write

$$
\begin{array}{l}
\mathrm{C\ L\ R\ L^3\ \overline{RL}^3\ R\ L\ R\ L^3\ R\ L\ L\ L} \ldots \\
\cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \ \ \cdot\ \ \ \overleftarrow{\mathrm{C}}\ \mathrm{L\ R\ L^3\ R\ L} \ldots
\end{array}
$$

that gives the augend $\mathrm{CLRL}^3\overline{\mathrm{RL}}^3$, and

$C\ L\ R\ L^3\ \overline{RL}^3\ R\ L\ R\ L\ L\ L\ R\ L^3\ R\ L\ L\ L \ldots$

$\cdot\ \cdot\ \cdot\ \cdot\ \ \cdot\ \ \cdot\ \cdot\ \cdot\ \cdot\ \overrightarrow{C}\ L\ R\ L^3\ R\ L \ldots$

that gives the augend $CLRL^3\overline{RL}^5$.

Therefore, we reach $(CLRL^3\overline{RL}^3R)LRL^2$ either by starting from the augend $CLRL^3\overline{RL}^3$ (of period 3×4 and equivalent to pattern CLR in $\mathbf{B}_0$) and by adding indefinitely $\overleftarrow{C}LRL^3RL$, or by starting from the augend $CLRL^3\overline{RL}^5$ (of period 4×4 and equivalent to pattern CLR^2 in $\mathbf{B}_0$) and by adding indefinitely $\overrightarrow{C}LRL^3RL$, as can be seen in Fig. 6.

As we saw in these examples, the inverse process can be more or less complex but it is always possible.

8. Conclusions

Misiurewicz points have already been studied by us [9,14,15] in a family of 1D quadratic maps, but it is here where we introduce a model of generation of the patterns of these points as we did in the case of the model of superstable periodic orbit patterns [25].

We show that if we add indefinitely a pattern (the addend) to another pattern (the augend), the resulting pattern has a preperiod and it is eventually periodic, that is, the pattern has the form as a Misiurewicz point pattern. Likewise, we saw that not all the patterns obtained in such a way correspond to a Misiurewicz point pattern but only those patterns where the augend is a superstable periodic orbit pattern and the addend is the augend itself or an heredity transmitter of the augend or a sum of heredity transmitters of the augend, that is, only the cases in which the heredity is taken into account.

In a family of 1D quadratic maps, every superstable periodic orbit has two characteristic Misiurewicz points associated, one on the left and one on the right. The patterns of these points are the result of adding indefinitely the first heredity transmitter to the superstable periodic orbit pattern, on the left and on the right. The characteristic Misiurewicz points corresponding to a superstable periodic orbit pattern have the property that within them all the descendants of this pattern and only these are present.

As we said before, the three only ways we have found to obtain a Misiurewicz point is by composing with itself, with an heredity transmitter or with a sort cut. Therefore, we had to do a study of short cuts, to introduce the concepts of basic composition, vectorial composition and direction basic composition, at the same time that we saw the equivalence and relation of short cuts.

We showed that simply by trial and error it is possible to develop the inverse process; that is, starting from a given Misiurewicz point pattern, it is possible to obtain the superstable periodic orbit pattern and the heredity transmitter that has to be indefinitely added to the first one. We have shown some examples to explain the procedure.

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Figure captions

Fig. 1. In the upper section, model of Misiurewicz points generation by composition of a superstable periodic orbit pattern with its heredity transmitters. In the lower part, a sketch of the corresponding zone of the Mandelbrot set antenna drawn by the escape lines method.

Fig. 2. In the upper section, model of Misiurewicz points generation by composition of a superstable periodic orbit pattern with short cuts. In the lower part, a sketch of the corresponding zone of the Mandelbrot set antenna drawn by the escape lines method.

Fig. 3. In the upper section, inverse process in the Misiurewicz points generation applied to two patterns whose periodic parts have different L parity. In the lower part, a sketch of the corresponding zone of the Mandelbrot set antenna drawn by the escape lines method.

Fig. 4. In the upper section, two examples of inverse process in the Misiurewicz points generation. The $(\mathrm{CLRL}^5)\mathrm{L}^3\mathrm{R}$ case show a model with related short cuts, and the $(\mathrm{CLRL}^6\mathrm{R})\mathrm{L}$ case show a model where a antiharmonic is involved. In the lower part, a sketch of the corresponding zone of the Mandelbrot set antenna drawn by the escape lines method.

Fig. 5. In the upper section, model of the inverse process in the Misiurewicz points generation applied to the $(\mathrm{CLR}^5)\mathrm{LR}^4\mathrm{LR}^2\mathrm{L}^2$ case. In the lower part, a sketch of the corresponding zone of the Mandelbrot set antenna drawn by the escape lines method.

Fig. 6. In the upper section, model of the inverse process in the Misiurewicz points generation applied to the $(\mathrm{CLRL}^3\overline{\mathrm{RL}}^3\mathrm{R})\mathrm{LRL}^2$ case, a Misiurewicz point pattern inside the chaotic band $\mathbf{B}_2$. In the lower part, a sketch of the corresponding zone of the Mandelbrot set antenna drawn by the escape lines method.