HOW TO WORK WITH ONE-DIMENSIONAL QUADRATIC MAPS*

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Abstract

We analyse some identifiers which can univocally identify hyperbolic components and Misiurewicz points of one-dimensional quadratic maps. After seeing the equivalence among the different identifiers and how to go from one to another, we show which are the best for some specific tasks. Likewise, we present the analytic expressions, some of them shown for the first time in this paper, to calculate these identifiers. Some experimental considerations are taken into account.

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1.- INTRODUCTION

In order to study non-linear phenomena, scientists from experimental sciences have the tool supplied by mathematicians: that of dynamical systems, whether continuous or discrete. When we work with discrete dynamical systems, the most used are the one-dimensional (1D) quadratic maps, being the most popular the logistic map $x_{n+1} = \lambda x_n (1 - x_n)$ and the real Mandelbrot map $x_{n+1} = x_n^2 + c$. The logistic map [1-3] is widely known among experimental scientists. Indeed, since Verhulst used it for the first time in 1845 to study population growth [1], it has served to model a large number of phenomena. The real Mandelbrot map is the intersection of the Mandelbrot set [4-6] and the real axis. Studies of 1D quadratic maps normally have a too mathematical language to be understood easily by experimental scientists. In this paper we shall try to give a less mathematical approach to the study of the 1D quadratic maps, so that the study can be within the experimental scientists' reach. So, the results presented in this paper are not rigorously proved theorems but the results of a vast quantity of data based on computer experiments.

1D quadratic maps are topologically conjugate [7-9]. Therefore, we can use one of them to study the others. To this end we normally use the real Mandelbrot map [10-12].

Let us study the two more representative elements of a 1D quadratic map: hyperbolic components [6] and Misiurewicz points [13-17]. There are many ways to name hyperbolic components and Misiurewicz points. For example, we can recognize a hyperbolic component by means of its period, and a Misiurewicz point by its preperiod and period. However, normally a lot of components have the same period, and a lot of Misiurewicz points have the same preperiod and period; hence, this way of naming them is not univocal. We are interested in names of hyperbolic components and Misiurewicz points that can identify them univocally. If this is the case, we denominate them "identifiers".

In our works with the real Mandelbrot map, we use as identifier the symbolic sequence [18], which is a sequence of the type CX…X (X can be a L for left, or a R for right) which shows the symbolic dynamics of the critical point in this map.

The real Mandelbrot map can be considered as the intersection of the Mandelbrot set and the real axis. Therefore, we can also identify hyperbolic components and Misiurewicz points of the real Mandelbrot map in the same way as we do for the Mandelbrot set. That is to say, another way to identify a hyperbolic component (or a Misiurewicz point) of a real Mandelbrot map is giving the external arguments associated to the external rays of Douady and Hubbard [14,19,20] that land in the cusp/root points of the cardioids/disks (or in the Misiurewicz points). These external arguments are given as rational numbers with odd denominator in the case of hyperbolic components, and with even denominator in the case of Misiurewicz points. These rational numbers can also be given as their binary expansions. Let us note that, in the case of the real Mandelbrot map, when we say "cardioids" and "disks" we really are referring to segments: the intersection of such cardioids and disks with the real axis.

Starting from the rational form of the external arguments, $\theta = \frac{a}{b}$, Milnor and Thurston obtained the kneading sequence [7], which is other type of symbolic dynamics. The kneading sequence can be considered another identifier; i.e. another way to name univocally hyperbolic components and Misiurewicz points of 1D quadratic maps.

Therefore, in the case of 1D quadratic maps, the univocal identification of hyperbolic components and Misiurewicz points can be accomplished by means of four identifiers. Two are sequences obtained by symbolic dynamics: the symbolic sequences and the kneading sequences, and another two are external arguments which can be given in two different ways: rational numbers and binary expansions.

These four identifiers are not the only possible ones. There are others but we only have given the more representative. For example, we normally name a Misiurewicz point as $M_{n,p}$, where *n* is the preperiod and *p* the period. So, $M_{8,5}$ is a Misiurewicz point with preperiod 8 and period 5. But there are 136 of such Misiurewicz points in a 1D quadratic map [16]. In order to name univocally each one of them, we can give an index *i*, $M_{8,5}^{(i)}$, that shows the appearance order when we cross the parameter values interval from $c = 0.25$ to $c = -2$ [16]. For example, $M_{8,5}^{(136)}$, is a Misiurewicz point with preperiod 8 and period 5, but is a specific one, the last one, among the 136 with the same preperiod and period; and therefore, is also an identifier.

In this paper we begin by showing, in the most didactically possible way, the equivalence among the four previous identifiers and how to obtain an identifier from the others. Next, we show some considerations about such identifiers. Finally, we show the expressions --some of them are given for the first time in this paper-- used to calculate hyperbolic components and Misiurewicz points, at the same time that we carried out a comparative analysis in order to know the best identifier for each calculus.

But, in addition, in those sections that need it, we shall use these identifiers in a particular experiment which behaves as a 1D quadratic map, in such a way that it may be easy to understand for engineers and experimental scientists.

There are innumerable phenomena that can be modelized by mean of 1D quadratic maps. We can find them not only in physics but also in all the experimental sciences. As an example, let us take the chemiluminescence of the reaction $CO+\frac{1}{2}O_2 \rightarrow CO_2$, that under some specific conditions is a chemical oscillator [21,22]. Indeed, at high pressure and temperature, in the presence of 0.5 % of H_2 an intermittent or oscillating ignition is obtained, in which we observe a period-doubling cascade followed by a chaotic region [21,22].

Let us see Fig. 1 obtained at a pressure of 20 Torr (see Fig. 6.11 in [21] or Fig. 7 in [22]). We start from 777.3 K and we slowly increase the temperature, which acts as a parameter. For a temperature interval each temperature corresponds to only one value of the ignition amplitude. But for a new interval, each temperature corresponds to two values of the ignition amplitude, oscillating from one value to the other. If we increase more the temperature, we reach another interval where the ignition amplitude has a period-4 oscillation for each temperature. Following on, we have new more and more little intervals where the ignition intensities have period-8, period-16, … oscillations for each temperature, although these values have such a little temperature interval that we cannot perceive it in this figure. Next, we reach the chaotic region where we also can see embedded into the chaos temperature intervals where the ignition amplitude oscillates periodically. If we increase again the temperature, we reach again a periodic region with a period-doubling cascade in the opposite direction. If we do not take into account this last opposite direction zone, this experiment behaves as a 1D quadratic map.

Insert Fig. 1

A 1D quadratic map is on a segment, therefore it is very difficult to visualize it. We pretend to “see” 1D quadratic maps so that we can work better with them. We can “see” 1D quadratic maps by using their bifurcation diagram given in Fig. 2(b). But Fig. 2(a), which is the neighbourhood of the real part of the Mandelbrot set, is equivalent to Fig. 2(b) (this equivalence was discussed in detail in Fig. 2 in [10]). Hence, we can also “see” a 1D quadratic map by using Fig. 2(a), although we know that only its intersection with the real axis is indeed a 1D quadratic map.

Insert Fig. 2

Fig. 3 is a sketch of a 1D quadratic map. If we start from the right end (cusp) and we go until the left end (tip), we cross two well differentiate regions: the periodic region or Feigenbaum region, which extends from the cusp to the Myrberg-Feigenbaum point MF, and the chaotic region which extends from MF to the tip. The periodic region is characterized by the period-doubling cascade, and the chaotic region by the chaotic bands doubling cascade. Each one of these chaotic bands have an infinity of hyperbolic components embedded in them, and they are separated by the merging points, which are Misiurewicz points. Hyperbolic components are periodic with period *p*, and they correspond to the ignition amplitude periodic oscillations of the experiment which happened in specific temperature intervals. To use, identify and operate with these hyperbolic components and Misiurewicz points of a 1D quadratic map would be of the most useful for experimental scientists.

Insert Fig. 3

2.- EQUIVALENCE AMONG IDENTIFIERS.

As we pointed out in the introduction, in order to identify univocally hyperbolic components and Misiurewicz points of a 1D quadratic map, we use the identifiers. We obtain these identifiers from either the symbolic dynamics (symbolic sequences [18] and kneading sequences [7]) or the external arguments of the external rays theory of Douady and Hubbard [14,19,20] (rational number and binary expansion). Hence, we have four identifiers:

- Identifiers
  - Sequences
    - symbolic sequence
    - kneading sequence
  - Arguments
    - rational number
    - binary expansion

In this section we shall see the equivalence among all these identifiers, and how to go from one to another (interchange). First we shall see the equivalence and the interchange between the two types of sequences (symbolic and kneading sequences). Next we shall see the same for the two types of external arguments (rational numbers and binary expansions). We shall finish by seeing the equivalences and interchanges between the sequences and the arguments.

2.1.- Equivalence between sequences (symbolic and kneading sequences)

Let us see the equivalence between symbolic sequences and kneading sequences for the case of hyperbolic components and for the case of Misiurewicz points.

2.1.1.- Hyperbolic components

The symbolic dynamics that we use, the symbolic sequence and the kneading sequence, are not only equivalent but in addition they are formally identical. Indeed, as we saw in the introduction, the symbolic sequence of a hyperbolic component of a 1D quadratic map is a sequence of the type CX…X (X is L for left or R for right) which shows the symbolic dynamics of the critical point in this map. According to MSS [23] a superstable orbit has a pattern (or symbolic sequence) with $p-1$ letters (L's and R's) properly combined. But a pattern corresponding to an orbit of period p with $p-1$ letters can be misleading and, to avoid that, some authors write the symbolic sequences with p letters by adding a C at the end [24] or at the beginning [25]. We use this last procedure to write symbolic sequences of hyperbolic components. Let us see, for example, the symbolic sequence of the period 7 hyperbolic component placed at $c=-1.674066091...$ that we shall see in Table 1 in section 3.1. The symbolic dynamics of the critical point is shown in Fig. 4. If we follow the orbit of $x_0=0$, and we assign a C for the centre, a L for the left and a R for the right, the symbolic sequence we obtain is: $\mathrm{CLRL^2RL}$.

Insert Fig. 4

As we also saw in the introduction, if we start from the rational form of the external arguments, $\theta=\frac{a}{b}$, that we shall see next, Milnor and Thurston obtain the kneading sequence [7]. The kneading sequence is the itinerary of θ when we apply 2θ mod 1. If, as can be seen in Fig. 5, it is in the open interval $\left(\frac{\theta}{2},\frac{\theta+1}{2}\right)$ we put A, and if it is in the open interval $\left(\frac{\theta+1}{2},\frac{\theta}{2}\right)$ we put B. If it is in $\frac{\theta}{2}$ or in $\frac{\theta+1}{2}$ we put *. (Lau and Schleicher [26] normally put 0 and 1 instead of A and B, but we prefer do not use 0 and 1 in the kneading sequence in order to do not make a mistake with the binary expansion of an external argument that also uses 0's and 1's).

Insert Fig. 5

Let us see the previous example, the period 7 hyperbolic component placed in $c = -1.674066091...$, whose first external argument is $\theta = \frac{54}{127}$. If we see the itinerary followed by 2θ mod 1, we have: $\theta_1 = \frac{54}{127}$, $\theta_2 = \frac{108}{127}$, $\theta_3 = \frac{89}{127}$, $\theta_4 = \frac{51}{127}$, $\theta_5 = \frac{102}{127}$, $\theta_6 = \frac{77}{127}$, $\theta_7 = \frac{27}{127}$, $\theta_8 = \frac{54}{127} = \theta_1$, …. We assign A to values into the open interval $\left(\frac{27}{127}, \frac{90.5}{127}\right)$. We assign B to values into the open interval $\left(\frac{90.5}{127}, \frac{27}{127}\right)$. We assign a * to values just in $\frac{27}{127}$ or $\frac{90.5}{127}$. Therefore, we have ABAABA*. But this sequence is the same obtained for the symbolic sequence if we change A=L, B=R and *=C (in our case, C was placed at the beginning; but, as we already noted in the introduction, other authors place it at the end, as * is placed). So, indeed, in the case of hyperbolic components the kneading sequence and the symbolic sequence are not only equivalent but formally identical.

Perhaps, this is why some authors name kneading sequence not only that obtained as the itinerary of θ when we apply 2θ mod 1, but also that obtained as the orbit of the critical point. That is to say, in the scientific literature some times both denominations (kneading and symbolic sequence) are indistinctly used.

2.1.2.- Misiurewicz points

As we know, Misiurewicz points $\mathrm{M}_{n,p}$ are preperiodic (preperiod *n*) and eventually periodic (period *p*). Normally we put the preperiod in brackets, $(\mathrm{CX}\cdots\mathrm{X})\mathrm{Y}\cdots\mathrm{Y}$, but sometimes ourselves, as other authors, represent it as $\mathrm{CX}\cdots\mathrm{X}\overline{\mathrm{Y}\cdots\mathrm{Y}}$. Like in the case of hyperbolic components, some authors do not write the C, and then the preperiod value decreases one unit. In this case we have $\mathrm{M}_{N,p}$, where the preperiod is $N = n - 1$. For example, let us see the Misiurewicz point $\mathrm{M}_{5,2}$, whose preperiod is 5 and whose period is 2, placed in $c = -1.430357632\ldots$, that we shall see in Table 2 in section 3.2. If we follow the orbit shown in Fig. 6, the symbolic sequence is: $\mathrm{CLRL}^2\overline{\mathrm{LR}}$ (the preperiodic orbit as a dashed line and the periodic orbit as a full line).

Insert Fig. 6

Let us see now the kneading sequence. The first of the external arguments of this Misiurewicz point is $\theta = \frac{99}{240}$. If we see the itinerary followed by 2θ mod 1, we have: $\theta_1 = \frac{99}{240}$, $\theta_2 = \frac{198}{240}$, $\theta_3 = \frac{156}{240}$, $\theta_4 = \frac{72}{240}$, $\theta_5 = \frac{144}{240}$, $\theta_6 = \frac{48}{240}$, $\theta_7 = \frac{96}{240}$, $\theta_8 = \frac{192}{240}$, $\theta_9 = \frac{144}{240} = \theta_5$,…. We assign A to values into the open interval $\left(\frac{49.5}{240},\frac{169.5}{240}\right)$. We assign B to values into the open interval $\left(\frac{169.5}{240},\frac{49.5}{240}\right)$. Hence, we have: $\mathrm{ABAA}\overline{\mathrm{AB}}$. But this sequence is the same obtained for the symbolic sequence if we change A=L, B=R (without C; that is to say, we have the form $\mathrm{M}_{N,p}$, where $N = 4$ and therefore $n = 5$). So, also in the Misiurewicz points case, the two sequences are not only equivalent but formally identical. Therefore, in this paper we only use one of them, the symbolic sequence, but taking into account that all we say about symbolic sequences is also valid for kneading sequences. As a summary of this section, we have:

$$\text{Kneading sequence} \Leftrightarrow \text{Symbolic sequence} \qquad \text{(E1)}$$

2.2.- Equivalence between the two types of external arguments

As we have already said, external arguments are normally given as rational numbers, but it is also frequent to give them as the binary expansions of such rational numbers. The binary expansion is another way of expressing the same, therefore it is obvious that both identifiers are equivalent. However, as we shall see later, they are not formally identical; so, in this paper we use both forms. Therefore, we have to see how to go from one to another. We shall see separately hyperbolic components and Misiurewicz point, beginning with the first ones and finishing with the second ones.

2.2.1.- Hyperbolic components

Let us begin by making some remarks about how to write external arguments as rational numbers. The only hyperbolic component of period 3 is $(\frac{3}{7},\frac{4}{7})$. As we shall see later in section 4.1., the first disk of its period doubling cascade is $H_F^{(1)}\left(\frac{3}{7},\frac{4}{7}\right) = \left(\frac{28}{63},\frac{35}{63}\right) = \left(\frac{4}{9},\frac{5}{9}\right)$. $\left(\frac{4}{9},\frac{5}{9}\right)$ is the *irreducible form* and $\left(\frac{28}{63},\frac{35}{63}\right)$ is the *normalized form* [27]. The normalized form is characterized because its denominator is $2^p - 1$, where p is the period of the hyperbolic component. If we rewrite the denominator by writing explicitly 2^p-1, we obtain what we name *explicit normalized form*, which in this case is: $\left(\frac{28}{2^6-1},\frac{35}{2^6-1}\right)$.

In the case of hyperbolic components it is very easy to go from an identifier to another. Let us consider an external argument whose identifier is given as a rational number in its explicit normalized form $\frac{a}{2^p-1}$, where *p* is the period, $a \le 2^p-1$, and both the numerator and the denominator are given in the decimal system. In order to go from the rational number to its binary expansion, we transform both the numerator and the denominator in the binary system. As we know from the transformation rules, the binary expansion of this external argument is $.\overline{a}$, obviously now *a* is given in binary and has to have *p* digits after the point. If we have not *p* digits when we transform *a* from decimal to binary, we add at the left the needing zeros up to have the *p* digits.

Example: Find the binary expansion of the previously cited period 7 hyperbolic component $\theta=\frac{54}{127}=\frac{54}{2^7-1}$. We transform 54 to binary and we have 110110; but, since $p=7$ we add one zero to the left and we have 0110110. Finally, the binary expansion is: $.\overline{0110110}$.

The inverse process --to go from the binary expansion to a rational number-- is also very easy. If we start from the binary expansion $.\overline{a}$, first we put it in the form $\frac{a}{2^p-1}$, where *a* is still in binary and *p* is the number of digits of the binary expansion. Next, we transform the numerator and the denominator to the decimal system.

Example: Find the rational number of the binary expansion $\theta=.\overline{0110110}$. First $.\overline{0110110}=\frac{110110}{2^7-1}$. Next, we transform the numerator and the denominator to the decimal system: $\theta=\frac{110110}{2^7-1}=\frac{54}{127}$.

2.2.2.- Misiurewicz points

The rational numbers corresponding to the external arguments of the Misiurewicz points have an even denominator, and can be given in the form $\frac{a}{(2^P-1)\cdot 2^N}$, where, as we shall see later, $N=n-1$ (being *n* the preperiod when C is taken into account, and *N* the preperiod when C is not taken into account) and *P* is either *p* or 2*p*, depending on the cases, being *p* the period. This is the *explicit normalized form*. Let us suppose we have the Misiurewicz point $\mathrm{M}_{6,1}$, with preperiod 6 and period 1, placed at $c = -1.697555393\ldots$, whose symbolic sequence is (CLRLLLR)L (or $\mathrm{CLRLLR}\overline{\mathrm{L}}$) (see Table 2 in section 3.2.) In this point, two external rays land (whose external arguments are $\left(\frac{41}{96}, \frac{55}{96}\right)$). As expected, the sum of the two external arguments equals 1. Therefore, we can use only one of them, the first one, because the second one can be directly deduced from it. $\frac{41}{96}$ can be expressed as $\frac{41}{96} = \frac{41}{3 \cdot 32} = \frac{41}{(2^2 - 1) \cdot 2^5}$. Hence $N = 5$. As $N = n - 1$, then $n = 6$. Likewise, here $P = 2p$ ($p = 1$).

Let us see how to go from the rational number to the binary expansion in the case of $\frac{41}{96}$.

$\frac{41}{96} = \frac{101001}{3 \cdot 32} = \frac{101001}{(2^2 - 1) \cdot 2^5} = \frac{101001}{1100000} = .01101\overline{10}$ (numerator and denominator are transformed to binary and next we do the binary division).

Let us see now the inverse process: going from the binary expansion to the rational number by starting from the previous binary expansion, $.01101\overline{10}$. As we know from mixed periodic fractions, we have:

$.011011\overline{01} = \frac{1101101 - 11011}{11000000} = \frac{101001}{1100000}$. Next, we transform the numerator and denominator to decimal system values: $\cdots = \frac{1 + 8 + 32}{32 + 64} =$ (or also $\cdots = \frac{1 + 8 + 32}{(2^2 - 1)2^5}$) $\cdots = \frac{41}{96}$.

In this section we have seen how to go from rational numbers to binary expansions and vice versa, for both hyperbolic components and Misiurewicz points, since both are equivalent. Therefore, we can write:

$$\text{Rational number} \Leftrightarrow \text{Binary expansion} \qquad \text{(E2)}$$

2.3.- Equivalence between the symbolic sequence and the binary expansion.

Each hyperbolic component and each Misiurewicz point have one symbolic sequence but two external arguments (except for the tip, that can be considered one double value). However, in the real case we are dealing with, the sum of the two external arguments of a hyperbolic component, or a Misiurewicz point, equals 1. Hence, given the binary expansion of one of the external arguments, we can calculate the other one simply by interchanging 0's and 1's. Therefore, it is enough to use only one of them; by convention, when we use only one, we use the external argument whose first digit after the point is 0. We shall begin by seeing hyperbolic components and afterwards Misiurewicz points.

2.3.1 .- Hyperbolic Components.

Let us see now that the binary expansion of an external argument of a hyperbolic component is equivalent to the symbolic sequence. In order to do so, we have to take into account the following points:

- The number of digits, after the point, of the binary expansion equals the number of letters of the symbolic sequence, which is the period *p* of the hyperbolic component.
- The n-th digit of the binary expansion represents the L-parity of the *n* first letters of the symbolic sequence [27].

If we take into account these two points, it is very easy to go from one to the other. Let us begin by going from the binary expansion to the symbolic sequence.

Let $.\overline{a}$ be the binary expansion of the external argument whose explicit normalized rational number is $\frac{a}{2^p - 1}$. Let us suppose that *a* begins by 0, because if it begins by 1 we change to the other argument which begins by 0, by interchanging all the 0's and 1's, as we saw before. The number of letters of the symbolic sequence is *p*, which coincides with the number of digits of the binary expansion. The first letter of the symbolic sequence is C which corresponds to the first digit, 0, of the binary expansion. The second digit shows the L-parity of the two first letters of the symbolic sequence. In the real case this second digit is always 1 (odd L-parity) therefore the first letter of the symbolic sequence after the C is L. The third digit shows the L-parity of the three first letters of the symbolic sequence. If it were 0 the L-parity would be even, therefore we should have CLL (this case never happens in 1D quadratic maps). If it is 1 the L-parity is odd, therefore we have CLR. The fourth digit shows the L-parity of the four first letters of the symbolic sequence. If it is 0 the L-parity is even, therefore we have to have CLRL. If it is 1 the L-parity is odd, therefore we have to have CLRR. And so on in such a manner that to see the n-th letter we have to see the n-th digit. If is 0, the L-parity of the *n* first letters is even. Therefore, if the L-parity of the n-1 previous letters were already even, we should add R, and if were odd, L. If, on the contrary, the n-th digit is 1, the L-parity of the *n* first letters is odd. If the L-parity of the n-1 previous letters were already odd, we should add R, and if were even, L.

Example: Find the symbolic sequence of $.\overline{01101}$

The first letter is C, which corresponds to the first digit, 0. The second is L because the L-parity of CL is odd (second digit 1). The third is R because the L-parity of CLR is odd (third digit 1). The fourth is L because the L-parity of CLRL is even (fourth digit 0). The fifth is L because the L-parity of CLRLL is odd (fifth digit 1). Hence, the symbolic sequence we have obtained is: CLRLL (or CLRL^2).

Next, let us see the inverse process; i. e., let us go from the symbolic sequence to the binary expansion. Let us start from the symbolic sequence $\mathrm{CX}_1 \cdots \mathrm{X}_{p\text{-}1}$, where X_i are either L or R. The binary expansion that we want find has to have *p* digits –the number of letters of the symbolic sequence- being the first of them by convention always 0, which corresponds to C. The second digit would be 1 if we had CL (odd L-parity) or 0 if we had CR (even L-parity). Since in the real case after the C we always have L, the second digit is necessarily 1. The third digit would be 0 if we had CLL (even L-parity) or 1 if we had CLR (odd L-parity). Since in the real case we always have CLR, the third digit is necessarily 1. The fourth digit would be 0 if we had CLRL (even L-parity) or 1 if we had CLRR (odd L-parity). And so on, in such a manner that the n-th digit is 0 if the n first letters of the symbolic sequence have even L-parity, or 1 if they have odd L-parity.

Example: Find the binary expansion of CLRL^2.

We begin by writing the symbolic sequence in its developed form; i. e., by writing its *p* letters, 5 letters in this case: CLRLL. By convention the first digit is 0, which corresponds to C. The second is 1 because the L-parity of CL is odd. The third is 1 because the L-parity of CLR is odd. The fourth is 0 because the L-parity of CLRL is even. The fifth is 1 because the L-parity of CLRLL is odd. Hence, the binary expansion is: $.\overline{01101}$.

The reader can use Table 1 in section 3.1. to practise in the conversion from symbolic sequences to binary expansions and vice versa.

2.3.2.- Misiurewicz points

Starting from the binary expansion of the external argument of a Misiurewicz point, we can obtain the symbolic sequence by taking into account the property that we have already seen for hyperbolic components: the n-th digit of the binary expansion shows the L-parity of the n first letters of the symbolic sequence.

Example: Find the symbolic sequence of $.01101\overline{10}$.

If we start from $.01101101010\cdots$, we obtain $\mathrm{CLRLLRLLL}\cdots \rightarrow \mathrm{CLRLLR}\overline{\mathrm{L}}$. Indeed, to 0 corresponds C. To the first 1 corresponds L (CL has odd L-parity). To the second 1 corresponds R (CLR has odd L-parity). And so on.

In order to carry out the inverse process, we have to take into account again the property of the L-parity.

Example: Find the binary expansion of $\mathrm{CLRL}^2\mathrm{R}\overline{\mathrm{L}}$.

Starting from $\mathrm{CLRLLR}\overline{\mathrm{L}} \rightarrow \mathrm{CLRLLRLLL}\cdots$, we obtain $.011011010101\cdots \rightarrow .01101\overline{10}$. Indeed, to C corresponds 1. To the first L corresponds 1 (CL has odd L-parity). To the first R corresponds another 1 (CLR has odd L-parity). And so on.

The reader can use Table 2 in section 3.2. to drill in both conversions.

As we can see from these examples and from Table 2 of section 3.2., the preperiod *n* of a Misiurewicz point symbolic sequence, written according to our method, has one unit more than the preperiod *N* of its binary expansion. Otherwise, in the case of 1D quadratic maps we are dealing with, the period *p* of a symbolic sequence can be either the period *P* of the binary expansion or the half *P*/2, as we shall see in section 3.2.

We have just seen how to go from the binary expansion to the symbolic sequence and vice versa, for both hyperbolic components and Misiurewicz points. Therefore, we can write:

$$\text{Binary expansion} \Leftrightarrow \text{Symbolic sequence} \qquad \text{(E3)}$$

2.4.- Equivalence between the rational number and the kneading sequence.

We have already seen that starting from the rational form of the external arguments, $\theta = \frac{a}{b}$, Milnor and Thurston obtain the kneading sequence [7], that is the itinerary of θ when we apply 2θ mod 1, for both hyperbolic components and Misiurewicz points.

The inverse process can be seen through the previous equivalences. That is to say, starting from the kneading sequence we obtain the symbolic sequence (see E1), from this one the binary expansion (see E3) and from this one the rational number (see E2). Hence, to sum up this last section we can write:

$$\text{Rational number} \Leftrightarrow \text{Kneading sequence} \qquad \text{(E4)}$$

If we take into account all the equivalences given, E1, E2, E3 and E4, we can give the following general diagram of equivalences:

$$
\begin{array}{ccc}
 & \text{(E1)} & \\
\textit{Kneading sequence} & \Leftrightarrow & \textit{Symbolic sequence} \\
\Updownarrow \text{(E4)} & & \Updownarrow \text{(E3)} \\
 & \text{(E2)} & \\
\textit{Rational number} & \Leftrightarrow & \textit{Binary expansion}
\end{array}
\qquad \text{(D1)}
$$

Therefore, in 1D quadratic maps these four identifiers (three if we consider that kneading sequences and symbolic sequences are identical) of hyperbolic components and Misiurewicz points are equivalent. That is so even though the two forms of a hyperbolic component external argument –rational number and binary expansion- refer to the root point (for disks) or to the cusp point (for cardioids), while the third form –symbolic sequence- refers to the centre of such a hyperbolic component. This identifiers equivalence is of the most importance for experimental scientists because they can use any of them in accordance with its own convenience.

A free computer program from W. Jung [28] is very useful to go from an identifier to another, and to calculate de centre parameter value of hyperbolic components.

3.- SOME CONSIDERATIONS ABOUT IDENTIFIERS.

Since the four identifiers are equivalent, we can use in principle whichever of them. However, depending on the specific tasks we have to accomplish, it may be better to use one identifier or another. Let us begin by hyperbolic components.

3.1.- Hyperbolic components

Insert Table 1

Table 1 shows 9 hyperbolic components. The first and second columns show periods and centres. The third, fourth and fifth columns show the identifiers. Indeed, the third column shows the symbolic sequence, with C at the beginning according to our method. We have not done a column for the kneading sequence because it is formally identical to the symbolic sequence. The fourth column shows the rational numbers of the external arguments. The fifth and last column gives the binary expansion of the same external arguments. Since the sum of the two external arguments equals 1, it is unnecessary to give both, because the second one can be easily obtained from the first one in both cases rational numbers and binary expansions.

What is the more suitable way of identifying a hyperbolic component? Well, we can say that there is no unique answer but for some cases it is better to use a given identifier and for other cases another one.

If we want a hyperbolic component identifier to be at the same time simple and as descriptive as possible, the symbolic sequence would have a slight advantage. Indeed, symbolic sequences are simple and very descriptive sequences, since the number of letters gives the period and the successive letters show the path followed by the orbit. The two forms of external arguments start with the handicap of being a couple for each hyperbolic component. Nevertheless, since the sum equals 1, we can consider only one and this handicap disappears. The binary expansion is simple as well, since we can also know directly the period by seeing the number of digits after the point. However, now we cannot know at first sight the path of the orbit. The rational number, as written in the fourth column, does not show at first sight the period, but if we rewrite it into the explicit normalized form, we could see the period at the first sight as well; though this form did not show the orbit path either.

Which of these identifiers shows better the position in the interval of the parameter values $-2 \le c \le 0.25$? In this case, the clear winner is the binary expansion because, since we only consider the first argument, all the values are arranged and included in the interval $(.\overline{0}, .0\overline{1})$. Therefore, given two any hyperbolic components, that which has the smaller value of its first argument binary expansion (what is seen at first sight) is on the right. For rational numbers, when we move from $c = 0.25$ to $c = -2$, values are also arranged in an increasing order from $(\frac{0}{1})$ to $(\frac{1}{2})$. Given two any hyperbolic components, that which has the smaller value of its first argument rational number is on the right (but now it can not be seen at first sight, because, in order to compare the two rational numbers, first we have to have common denominator). In this case, the symbolic sequence has the worst behaviour. Given a symbolic sequence, it is difficult to foresee its position, unless one is very experienced in using symbolic sequences.

3.2.- Misiurewicz points

Insert Table 2

Table 2 shows 11 Misiurewicz points. The first column shows preperiods and periods, and the second column shows the localization. The other three columns show the identifiers: the third column shows the symbolic sequence, with C at the beginning and with the preperiod sequence in brackets, in accordance with our method (let us remember that most authors do not write the C, and then their preperiods, *N*, has one unit less than our preperiod, *n*; i. e., $N = n-1$). We have not here put a column for the kneading sequence either because again it is formally identical to the symbolic sequence. The fourth and fifth columns show the two external arguments of the corresponding Misiurewicz points, the rational numbers in the fourth one and the binary expansions in the fifth one. As in the hyperbolic components case the sum of the two external arguments in both forms equals 1, and therefore it is unnecessary to give both, and the second one can be easily obtained from the first one.

By taking into account Table 2, let us see which is the more suitable way to identify a Misiurewicz point. If we want the identifier to be both simple and descriptive, again the winner is the symbolic sequence because the number of letters in bracket, *n*, is the preperiod and the number of letters out of brackets, *p*, is the period. Likewise, the successive letters show here again the path followed by the orbit. Rational numbers, as written in the fourth column, do not show at first sight either the preperiod or the period; but if we rewrite it in the explicit normalized form, $\frac{a}{(2^P - 1)\cdot 2^N}$, we could obtain more information. Indeed, the preperiod according to our method is $n = N+1$. We do not know the period right away, but we know that is either *P* or *P*/2. The binary expansion is a little more simple than rational numbers, because we do not need rewrite anything, but not so simple as the symbolic sequence. The preperiod can also be directly known since $n = N+1$, where now *N* is the number of digits after the point. In order to see the period, we analyze the periodic digits, where we can find two cases: (a) If we have a different number of 0's and 1's, then the period is the number of periodic digits, $p = P$. (b) If we have the same number of 0's and 1's, (and in addition the semisequences of periodic digits are complementary) then the period is the half of the number of periodic digits, $p = \frac{P}{2}$. For example, the last but one Misiurewic point in Tale 2, placed in $c = -1.423729232\ldots$, has 12 non-periodic digits, then the preperiod according to our method is 13. It has 8 periodic digits with the same number of 0's and 1's. In addition, the periodic semisequences are 1001 and 0110, which are complementary. Hence, the period is 4. Nevertheless, the last Misiurewicz point in Table 12, placed in $c = -1.996548203\ldots$, has 6 non periodic digits, then our preperiod is 7. It has 10 periodic digits, and since the number of 0's and 1's is different, the period is also 10. The reader can train with the other values of Table 2.

As it happens with hyperbolic components, the binary expansion is the identifier that better shows the position in the parameter values interval $-2 \leq c \leq 0.25$. If we only consider the first argument as usually, again all the values are arranged and included in the interval $(.\overline{0}, .0\overline{1})$. Therefore, given two any Misiurewicz points, that which has the smaller value of the first argument binary expansion is on the right. Likewise, given two any Misiurewicz points as rational numbers, that which has the smaller value of its first argument rational number is on the right (but now it can not be seen at first sight; because, in order to compare the two rational numbers, first we have to have common denominator). Again the symbolic sequence has the worst behaviour because it is not easy to foresee its position.

3.3. Applications.

Let us consider again the experiment where CO and oxygen reacts to give CO_2, shown in the introduction. Let us suppose we can detect the intermittence (oscillation) of the ignition amplitude for the following periodic values: 1, 2 and 4, … in the periodic region (period doubling cascade), and 6, 5, 3, 6, 5, 6, 4, 6, 5 and 6 (in this order when the temperature increase) in the chaotic region. In Fig. 7 we show again the 1D quadratic map of the Fig. 3 with the hyperbolic components corresponding to the previous experimental data (plus a period-7 hyperbolic component and the merging point m_1). As we can see, there are several temperature intervals with the same oscillation period (cases of periods 5, 4 or 6). Therefore, we have to differentiate them with identifiers.

Insert Fig. 7

In Fig. 7 we show the three identifiers for each one of the hyperbolic components (temperature intervals with the same oscillation period) detected in the experiment. Let us note that when a hyperbolic component has two identifiers we only give one, the smaller. The advantage of using the binary expansion as identifier is that its ordering can be seen with the naked eye; so, for two given values, the one that has greater binary expansion occurs at greater temperature. We can say the same of the rational form, but now we cannot see the ordering with the naked eye. (Let us note that in Figs. 2, 3 and 7 we go forward from the right to the left, while in Fig. 1 the temperature increases from the left to the right).

The advantage of the symbolic sequence is that L’s and R’s give an idea of the ignition amplitude oscillation. If C is the ignition amplitude of the critical point, R’s correspond to a value greater than C, and L’s to a value smaller than C.

Since some identifiers present some advantages and other identifiers others, and since all the identifiers are equivalent, the experimental scientist may choose the more useful identifier for each specific case.

4.- OPERATING WITH IDENTIFIERS

We have tried hard to “calculate” hyperbolic components and Misiurewicz points; i.e. to introduce analytical expressions that allows one to find their identifiers. In this section we shall revise these analytic expressions which in the most part were introduced by us in a previous paper [27], though we present new ones in this paper. But above all we shall see the better identifier in any specific case.

4.1.- Hyperbolic components.

We shall see first the MSS harmonics used to calculate the identifiers of the period doubling cascade disks of a given hyperbolic component. These MSS harmonics were introduced by Metropolis, Stein and Stein for the case of symbolic sequences [23], and we gave the analytical expressions for the two cases of external arguments [27]. If the hyperbolic component is given by its explicit normalized rational numbers, $\left(\frac{a_1}{2^p-1},\frac{a_2}{2^p-1}\right)$, then the order i MSS harmonic is:

$$H_{MSS}^{(i)}\left(\frac{a_1}{2^p-1},\frac{a_2}{2^p-1}\right)=\left(\frac{a_{1(i-1)}2^{2^{i-1}p}+a_{2(i-1)}}{2^{2^i p}-1},\frac{a_{2(i-1)}2^{2^{i-1}p}+a_{1(i-1)}}{2^{2^i p}-1}\right),\quad i\geq 1 \tag{1}$$

where $a_{1(0)}=a_1$ and $a_{2(0)}=a_2$.

If the hyperbolic component is given by the binary expansions, $\left(.\overline{a_1},.\overline{a_2}\right)$, then the successive MSS harmonics are:

$$H_{MSS}^{(1)}\left(.\overline{a_1},.\overline{a_2}\right)=\left(.\overline{a_1a_2},.\overline{a_2a_1}\right)$$

$$H_{MSS}^{(2)}\left(.\overline{a_1},.\overline{a_2}\right)=\left(.\overline{a_1a_2a_2a_1},.\overline{a_2a_1a_1a_2}\right)$$

$$H_{MSS}^{(3)}\left(.\overline{a_1},.\overline{a_2}\right)=\left(.\overline{a_1a_2a_2a_1a_2a_1a_1a_2},.\overline{a_2a_1a_1a_2a_1a_2a_2a_1}\right) \tag{2}$$

$$\ldots,$$

If the hyperbolic component is given by its symbolic sequence, the first harmonic (first disk of the period doubling cascade) is calculated by composing the symbolic sequence with itself by using the left composition rule (lero rule) [12]. If the obtained result is composed with itself by using again the lero rule, we obtain the second harmonic (second disk of the period doubling cascade), and so on. As we have just seen, the more complex form of calculating the MSS harmonics is by using rational numbers, and the more simple is by using binary expansions.

Let us see next the Fourier harmonics (F-harmonics or simply harmonics) of a hyperbolic component which, as we know, except for the first, are used to calculate the last appearance hyperbolic components of the corresponding chaotic band [12]. These F-harmonics have been introduced by us for both symbolic sequences [12] and external arguments [27].

If the hyperbolic component is given by the explicit normalized rational numbers, $\left(\frac{a_1}{2^p-1},\frac{a_2}{2^p-1}\right)$, the order i F-harmonic is:

$$H_F^{(i)}\left(\frac{a_1}{2^p-1},\frac{a_2}{2^p-1}\right)=\left(\frac{a_1 2^{ip}+a_2\sum_{j=0}^{j=i-1}2^{jp}}{2^{(i+1)p}-1},\frac{a_2 2^{ip}+a_1\sum_{j=0}^{j=i-1}2^{jp}}{2^{(i+1)p}-1}\right) \tag{3}$$

If the hyperbolic component is given by the binary expansions, $\left(.\overline{a_1},.\overline{a_2}\right)$, the order i F-harmonic is:

$$H_F^{(i)}\left(.\overline{a_1},.\overline{a_2}\right)=\left(.\overline{a_1\underbrace{a_2a_2\cdots a_2}_{i}},.\overline{a_2\underbrace{a_1a_1\cdots a_1}_{i}}\right) \tag{4}$$

Let $P=CQ$ be the symbolic sequence of a period p hyperbolic component. Q is a sequence of L's and R's with p-1 letters. We calculate the order i F-harmonic by composing the symbolic sequence with itself i times, by using the lero rule [12]. So:

$$H_F^{(i)}\left(CQ\right)=CQ\underbrace{\overleftarrow{C}Q\overleftarrow{C}Q\cdots\overleftarrow{C}Q}_{i} \tag{5}$$

Again, the more complex form of calculating F-harmonics is using rational numbers and the simpler, binary expansions.

Finally, let us see the leftward and rightward compositions used to calculate the descendants of a hyperbolic component [29].

Let $\left(\frac{a_1}{2^p-1},\frac{a_2}{2^p-1}\right)$ and $\left(\frac{a'_1}{2^{p'}-1},\frac{a'_2}{2^{p'}-1}\right)$ be the hyperbolic components of the augend and addend which are given by the explicit normalized rational numbers. The leftward and rightward compositions are:

$$\left(\frac{a_1}{2^p-1},\frac{a_2}{2^p-1}\right)\overleftarrow{+}\left(\frac{a'_1}{2^{p'}-1},\frac{a'_2}{2^{p'}-1}\right)=\left(\frac{2^{p'}a_1+a'_2}{2^{(p+p')}-1},\frac{2^{p'}a_2+a'_1}{2^{(p+p')}-1}\right),$$

$$\left(\frac{a_1}{2^p-1},\frac{a_2}{2^p-1}\right)\overrightarrow{+}\left(\frac{a'_1}{2^{p'}-1},\frac{a'_2}{2^{p'}-1}\right)=\left(\frac{2^{p'}a_1+a'_1}{2^{(p+p')}-1},\frac{2^{p'}a_2+a'_2}{2^{(p+p')}-1}\right) \tag{6}$$

Let $\left(.\overline{a_1},.\overline{a_2}\right)$ and $\left(.\overline{a'_1},.\overline{a'_2}\right)$ be the hyperbolic components of the augend and addend which are given by the binary expansions. The leftward and rightward compositions are:

$$\left(.\overline{a_1},.\overline{a_2}\right)\overleftarrow{+}\left(.\overline{a'_1},.\overline{a'_2}\right)=\left(.\overline{a_1a'_2},.\overline{a_2a'_1}\right)$$

$$\left(.\overline{a_1},.\overline{a_2}\right)\overrightarrow{+}\left(.\overline{a'_1},.\overline{a'_2}\right)=\left(.\overline{a_1a'_1},.\overline{a_2a'_2}\right) \tag{7}$$

Let $P_1 = CQ_1$ and $P_2 = CQ_2$ be the hyperbolic components of the augend and addend which are given by the symbolic sequences. The leftward and rightward compositions are:

$$CQ_1 \overleftarrow{+} CQ_2 = CQ_1\overleftarrow{C}Q_2$$

$$CQ_1 \overrightarrow{+} CQ_2 = CQ_1\overrightarrow{C}Q_2 \tag{8}$$

i. e. in the leftward composition we apply the lero rule and in the rightward composition the relo rule [12].

As in the other cases, the more complex way of calculating the leftward and rightward compositions is using rational numbers and the simpler using binary expansions.

Summarizing, when we calculate hyperbolic components, the binary expansion offers in general more advantages. However, since it is very easy to go from one identifier to another, there is no problem if we work with other identifiers that can be advantageous in specific cases.

4.2.- Misiurewicz points

Let us see how to calculate Misiurewicz points.

The order ∞ F-harmonic of a cardioid is the tip of such a cardioid, while the order ∞ F-harmonic of a disk of the period doubling cascade is a merging point [27]. Both, tips and merging points, are Misiurewicz points. Let us see how to calculate these order ∞ F-harmonics in each one of the three identifiers.

If the cardioid or the disk are given by their explicit normalized rational numbers, $\left(\frac{a_1}{2^p-1},\frac{a_2}{2^p-1}\right)$, the order ∞ F-harmonic is [27]:

$$H_F^{(\infty)}\left(\frac{a_1}{2^p-1},\frac{a_2}{2^p-1}\right)=\left(\frac{a_1\left(2^p-1\right)+a_2}{2^p\left(2^p-1\right)},\frac{a_1+a_2\left(2^p-1\right)}{2^p\left(2^p-1\right)}\right) \tag{9}$$

If the cardioid or the disk are given by their binary expansions, $\left(.\overline{a_1},.\overline{a_2}\right)$, the order ∞ F-harmonic is [27]:

$$H_F^{(\infty)}\left(.\overline{a_1},.\overline{a_2}\right)=\left(.a_1\,\underbrace{\overline{a_2 a_2\cdots a_2}}_{\infty},.a_2\,\underbrace{\overline{a_1 a_1\cdots a_1}}_{\infty}\right)=(.a_1\overline{a_2},.a_2\overline{a_1}) \tag{10}$$

Let $P=CQ$ be the symbolic sequence of the cardioid or disk whose period is *p*. We calculate the order ∞ F-harmonic by composing the symbolic sequence with itself ∞ times by using the lero rule [12]. That is to say:

$$H_F^{(\infty)}\left(CQ\right)=CQ\underbrace{\overleftarrow{C}Q\overleftarrow{C}Q\cdots\overleftarrow{C}Q}_{\infty} \tag{11}$$

Also here the more complex way to calculate F-harmonics is using rational numbers, and the simpler using binary expansions. Let us see finally the leftward composition and the rightward composition, applied ∞ times, which are used to calculate Misiurewicz points in general [17].

Let $\left(\frac{a_1}{2^p-1},\frac{a_2}{2^p-1}\right)$ be the explicit normalized rational numbers of the external arguments of a hyperbolic component and $\left(\frac{a_1'}{2^{p'}-1},\frac{a_2'}{2^{p'}-1}\right)$ the explicit normalized rational numbers of the external arguments of a heredity transmitter ancestor [29]. Infinite leftward compositions of the heredity transmitter ancestor give the external arguments of a Misiurewicz point located on the left of the hyperbolic component. As it is easy to prove mathematically, the former infinite repetition of the leftward composition gives [27]:

$$\left[\left[\left(\frac{a_1}{2^p-1},\frac{a_2}{2^p-1}\right)\overleftarrow{+}\left(\frac{a_1'}{2^{p'}-1},\frac{a_2'}{2^{p'}-1}\right)\right]\overleftarrow{+}\left(\frac{a_1'}{2^{p'}-1},\frac{a_2'}{2^{p'}-1}\right)\right]\overleftarrow{+}\ldots=\left(\frac{a_1+\dfrac{a_2'}{2^{p'}-1}}{2^p},\frac{a_2+\dfrac{a_1'}{2^{p'}-1}}{2^p}\right) \tag{12}$$

In the same way, infinite rightward compositions give the external arguments of a Misiurewicz point located on the right of the hyperbolic component. Likewise, the infinite repetition of the rightward composition gives [27]:

$$\left[\left[\left(\frac{a_1}{2^p-1},\frac{a_2}{2^p-1}\right)\overrightarrow{+}\left(\frac{a_1'}{2^{p'}-1},\frac{a_2'}{2^{p'}-1}\right)\right]\overrightarrow{+}\left(\frac{a_1'}{2^{p'}-1},\frac{a_2'}{2^{p'}-1}\right)\right]\overrightarrow{+}\ldots=\left(\frac{a_1+\dfrac{a_1'}{2^{p'}-1}}{2^p},\frac{a_2+\dfrac{a_2'}{2^{p'}-1}}{2^p}\right) \tag{13}$$

Let $\left(.\overline{a_1},.\overline{a_2}\right)$ be the binary expansion of the external arguments of a hyperbolic component and $\left(.\overline{a_1'},.\overline{a_2'}\right)$ the binary expansion of the external arguments of a heredity transmitter ancestor [29]. Infinite leftward compositions of the heredity transmitter ancestor give the external arguments of a Misiurewicz point located on the left of the hyperbolic component. As it is easy to prove mathematically, the former infinite repetition of the leftward composition gives:

$$\left[\left[\left(.\overline{a_1}, .\overline{a_2}\right) \overleftarrow{+} \left(.\overline{a'_1}, .\overline{a'_2}\right)\right] \overleftarrow{+} \left(.\overline{a'_1}, .\overline{a'_2}\right)\right] \overleftarrow{+} \ldots = \left(.\overline{a_1 \underbrace{a'_2 \cdots a'_2}_{\infty}}, .\overline{a_2 \underbrace{a'_1 \cdots a'_1}_{\infty}}\right) = \left(.a_1\overline{a'_2}, .a_2\overline{a'_1}\right) \qquad (14)$$

In the same way, infinite rightward compositions give the external arguments of a Misiurewicz point located on the right of the hyperbolic component. Likewise, the infinite repetition of the rightward composition gives:

$$\left[\left[\left(.\overline{a_1}, .\overline{a_2}\right) \overrightarrow{+} \left(.\overline{a'_1}, .\overline{a'_2}\right)\right] \overrightarrow{+} \left(.\overline{a'_1}, .\overline{a'_2}\right)\right] \overrightarrow{+} \ldots = \left(.\overline{a_1 \underbrace{a'_1 \cdots a'_1}_{\infty}}, .\overline{a_2 \underbrace{a'_2 \cdots a'_2}_{\infty}}\right) = \left(.a_1\overline{a'_1}, .a_2\overline{a'_2}\right) \qquad (15)$$

Formulae (14) and (15) have been obtained in this paper for the first time.

Let $P_1 = CQ_1$ be the symbolic sequence of a hyperbolic component and let $P_2 = CQ_2$ be the symbolic sequence of a heredity transmitter ancestor of P_1. Infinite leftward compositions of the heredity transmitter ancestor give the symbolic sequence of a Misiurewicz point located on the left of the hyperbolic component [12].

$$CQ_1 \overleftarrow{+} \underbrace{CQ_2 \overleftarrow{+} CQ_2 \overleftarrow{+} \cdots \overleftarrow{+} CQ_2}_{\infty} = CQ_1 \underbrace{\overleftarrow{C}Q_2 \overleftarrow{C}Q_2 \cdots \overleftarrow{C}Q_2}_{\infty} \qquad (16)$$

In the same way, infinite rightward compositions of the heredity transmitter ancestor give the symbolic sequence of a Misiurewicz point located on the right of the hyperbolic component [12].

$$CQ_1 \overrightarrow{+} \underbrace{CQ_2 \overrightarrow{+} CQ_2 \overrightarrow{+} \cdots \overrightarrow{+} CQ_2}_{\infty} = CQ_1 \underbrace{\overrightarrow{C}Q_2 \overrightarrow{C}Q_2 \cdots \overrightarrow{C}Q_2}_{\infty} \qquad (17)$$

Here also, like in hyperbolic components, the more complex way to calculate Misiurewicz points is using rational numbers, and the simpler using binary expansions according to eqs (14) and (15), both given in this paper for the first time.

4.3. Applications.

By using MSS harmonics we can calculate the discs of the period-doubling cascade of the periodic region. Even if the experiment that we use as an example only could detect temperature intervals where the ignition amplitude oscillates with periods 1, 2 and 4, we know that next there are intervals where the ignition amplitude oscillates with periods 8, 16, …, which cannot be detected, for example, for lack of experimental resolution. However, the previously introduced formulae give us some information about them. For example, the third MSS harmonic of the main cardioid is the period-8 disc $H_{MSS}^{(3)}\left(.\overline{0},.\overline{1}\right)=\left(.\overline{01101001},.\overline{10010110}\right)$ (we only take into account the first value). As we can see in Fig. 7, it should be placed next to the period-4 interval, what we already knew. In this case it is more interesting to consider the symbolic sequence: $H_{MSS}^{(3)}\left(\mathrm{C}\right)=\left(\mathrm{CLRL^3RL}\right)$. Now we not only know the oscillation period, we also know the orbit: i. e., the ordering of the values above C (R's) or under C (L's). Knowing the orbit can help to detect a periodic interval that is in the limit of the experimental resolution.

Let us see now the chaotic region. As we saw before, we consider that the experimental conditions only allow to detect temperature intervals at most with period-6 oscillations. Let us suppose we are interested in detecting a period-7 orbit, not detected at first. Our formulae make the work easier, and help the search, although it does not guarantee its success. Let us look for the period-7 orbit shown in Fig. 7. As we saw before, the three identifiers of such a period-7 hyperbolic component are calculated as follows (in the two last identifiers we only use the first one): $CLRL^2 \overrightarrow{+} CL = CLRL^4$; $\left(\frac{13}{31},\frac{18}{31}\right)\overrightarrow{+}\left(\frac{1}{3},\frac{2}{3}\right)=\left(\frac{53}{127},\frac{74}{127}\right)$ and $\left(.\overline{01101},.\overline{10010}\right)\overrightarrow{+}\left(.\overline{01},.\overline{10}\right)=\left(.\overline{0110101},.\overline{1001010}\right)$.

The values of the rational form, and above all those of the binary expansion, allow us to know the zone where we have to search. So, in the case we are seeing, since $.\overline{011010}<.\overline{0110101}<.\overline{01101}$ (or $\frac{23}{63}<\frac{53}{127}<\frac{13}{31}$), the searched period-7 orbit has to be between the first period-6 orbit and the first period-5 orbit when we increase the temperature from MF. This is already an advantage because it is not the same to look in the whole range of temperatures than to look in a small interval. But, in addition, the knowledge of the symbolic sequence shows us the orbit: i. e., the form of the oscillation, that can help to the success of our searching, because we can more easily detect an oscillation if we know its form.

The merging points are Misiurewicz points, which are unstable points, and therefore it is very difficult to detect them experimentally. But we can calculate them and know their situation in the experiment. Indeed, if we calculate the binary expansion of a merging point, all the orbits whose binary expansion are smaller, are in the right chaotic band; and all the orbits whose binary expansion are greater, are in the left chaotic band. Thus, m_1, the merging point of the chaotic bands $\mathbf{B_0}$ and $\mathbf{B}_1$, is $m_1=H_F^{(\infty)}\left(.\overline{01},.\overline{10}\right)=\left(.01\overline{10},.10\overline{01}\right)$. Hence, the first period-6 orbit is placed on the right of m_1 (chaotic band $\mathbf{B}_1$) because $.01\overline{10} > .\overline{011010}$ and all the other orbits are placed on the left (chaotic band $\mathbf{B}_0$) because $.01\overline{10} < .\overline{0110101}$ (see Fig. 7).

5.- CONCLUSIONS.

Hyperbolic components and Misiurewicz points of 1D quadratic maps can be identified in many ways. For example, by giving the periods of the hyperbolic components and the preperiods and periods of the Misiurewicz points. However, this way of identification is not univocal. If we want the identification to be univocal, we have to use the identifiers.

We analyse here four identifiers, two supplied by the symbolic dynamics (symbolic sequences and kneading sequences) an another two by the external arguments of the external rays theory of Douady and Hubbard (rational numbers and binary expansions), although here we only analyse three because the two first are formally identical.

We show that identifiers are equivalent, and how to go from an identifier to another. Since identifiers are equivalent, we can use any of them; however, in most of the cases the more advantageous is to use the binary expansion.

We present the analytical expressions to calculate the identifiers of hyperbolic components and Misiurewicz points. Although some of these expressions have already been introduced by us, some of them have been introduced by the first time in this paper. We fulfil a comparative study and we obtain again the binary expansion as the most advantageous identifier.

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Figure Captions

Fig. 1 Ignition amplitude oscillations made by allowing the temperature to increase at fixed high pressure ($p = 20\,Torr$). A region of period-doubling cascade oscillations (periodic region) and a region of chaotic oscillations with periodic ones embedded in them (chaotic region) can be observed.

Fig. 2 (a) Real part of the Mandelbrot set $z_{n+1} = z_n^2 + c$; (b) bifurcation diagram of the real Mandelbrot map $x_{n+1} = x_n^2 + c$.

Fig. 3 A sketch "to see" a 1D quadratic map: the neighbourhood of the real part of the Mandelbrot set (only the intersection with the real axis is a 1D quadratic map).

Fig. 4 Superestable orbit $\mathrm{CLRL^2RL}$ for the parameter value $c = -1.674066091...$

Fig. 5 A sketch to find the kneading sequence corresponding to the external argument θ, after applying 2θ mod 1.

Fig. 6 Preperiodic (dashed line) and eventually periodic (full line) orbit $\mathrm{(CLRL^2)LR}$ for the parameter value $c = -1.430357632...$

Fig. 7 A sketch of the 1D quadratic map corresponding to the experimental data. A label with the three identifiers of each periodic oscillation is shown.

Table Captions

Table 1 Periods, centre parameters and identifiers of nine hyperbolic components.

Table 2 Preperiods and periods, parameters and identifiers of eleven Misiurewicz points.

Title: How to work with one-dimensional quadratic maps. Authors: G. Pastor et al.

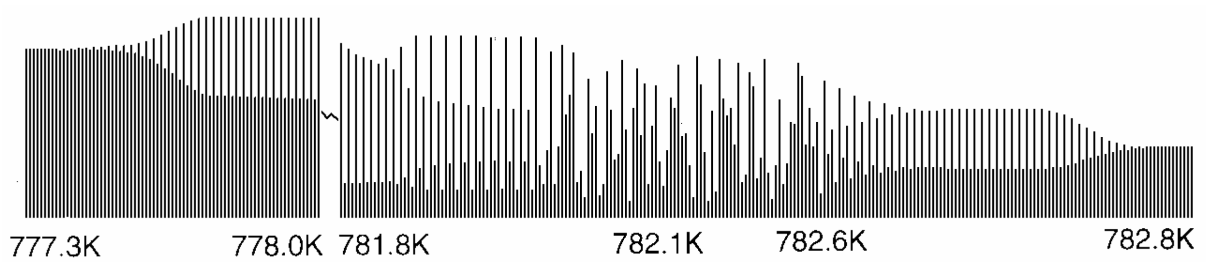

Fig.1

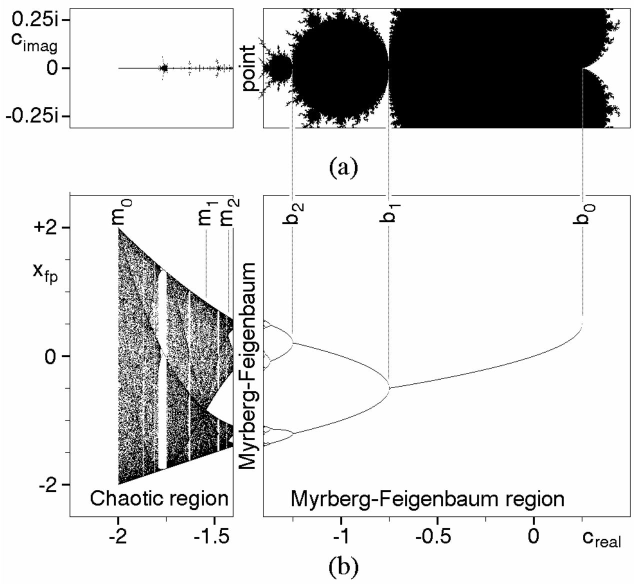

Fig. 2

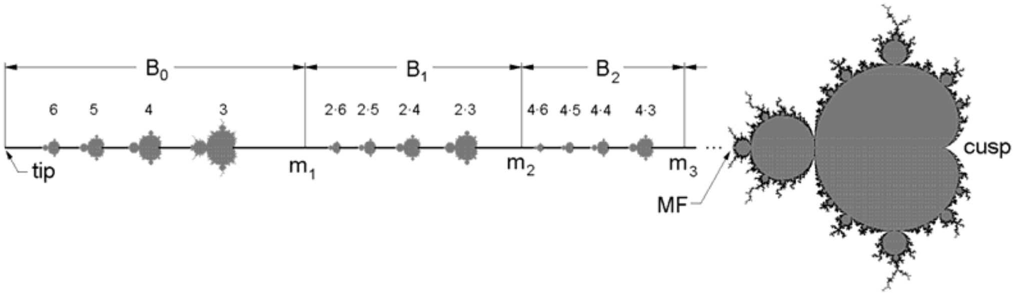

Fig. 3

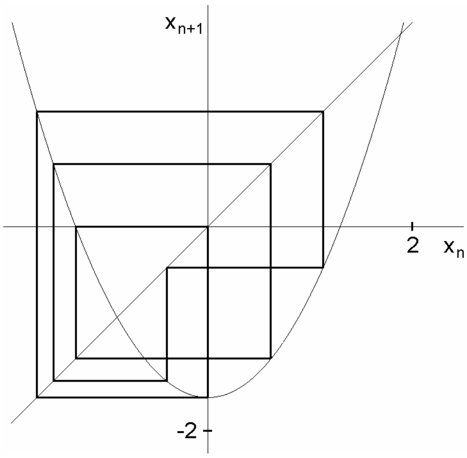

Fig. 4.

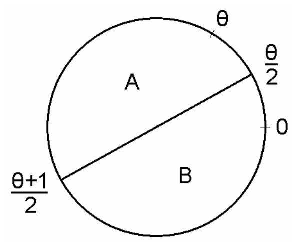

Fig. 5

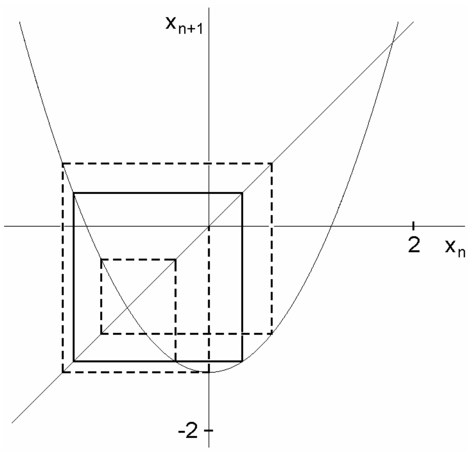

Fig 6

Title: How to work with one-dimensional quadratic maps.   Authors: G. Pastor et al.

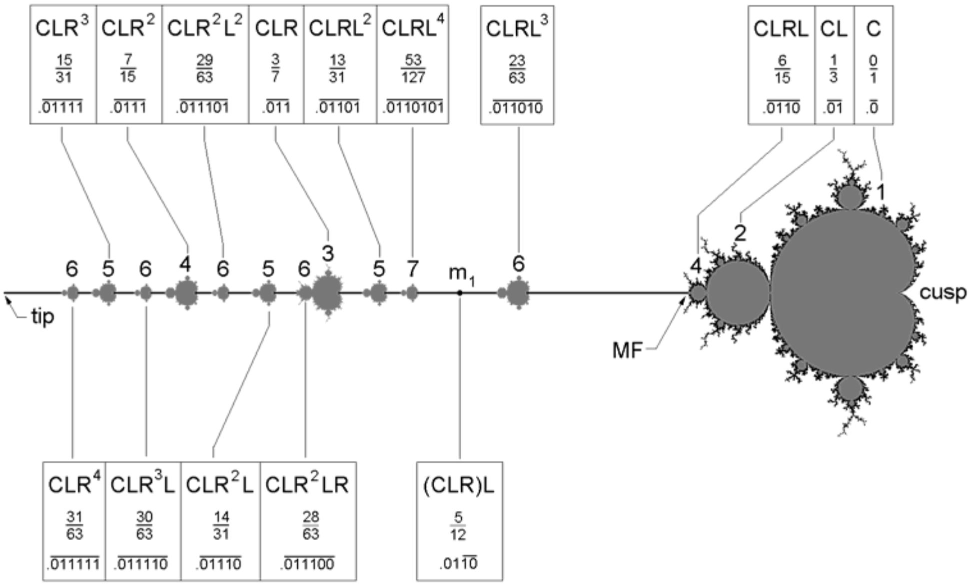

Fig. 7

Title: How to work with one-dimensional quadratic maps.   Authors: G. Pastor et al.

Period p	Parameter c	Symbolic sequence	External arguments (rational form)	External arguments (binary expansion form)
1	0	C	$\left(\frac{0}{1}, \frac{1}{1}\right)$	$\left(.\overline{0}, .\overline{1}\right)$
2	-1	CL	$\left(\frac{1}{3}, \frac{2}{3}\right)$	$\left(.\overline{01}, .\overline{10}\right)$
3	$-1.754877666\ldots$	CLR	$\left(\frac{3}{7}, \frac{4}{7}\right)$	$\left(.\overline{011}, .\overline{100}\right)$
5	$-1.625413725\ldots$	$CLRL^2$	$\left(\frac{13}{31}, \frac{18}{31}\right)$	$\left(.\overline{01101}, .\overline{10010}\right)$
7	$-1.674066091\ldots$	$CLRL^2RL$	$\left(\frac{54}{127}, \frac{73}{127}\right)$	$\left(.\overline{0110110}, .\overline{1001001}\right)$
9	$-1.993130254\ldots$	CLR^4L^3	$\left(\frac{250}{511}, \frac{261}{511}\right)$	$\left(.\overline{011111010}, .\overline{100000101}\right)$
11	$-1.972824987\ldots$	CLR^3L^5R	$\left(\frac{980}{2047}, \frac{1067}{2047}\right)$	$\left(.\overline{01111010100}, .\overline{10000101011}\right)$
13	$-1.892757718\ldots$	CLR^2L^9	$\left(\frac{3754}{8191}, \frac{4437}{8191}\right)$	$\left(.\overline{0111010101010}, .\overline{1000101010101}\right)$
19	$-1.612331876\ldots$	$CLRL^4RL^6RL^4$	$\left(\frac{219829}{524287}, \frac{304458}{524287}\right)$	$\left(.\overline{0110101101010110101}, .\overline{1001010010101001010}\right)$

Table 1

Title: How to work with one-dimensional quadratic maps.   Authors: G. Pastor et al.

Preperiod and period *n*,*p*	Parameter *c*	Symbolic sequence	External arguments (rational form)	External arguments (binary expansion form)
2,1	–2	(CL)R	$\left(\frac{1}{2},\frac{1}{2}\right)$	$(.1,\ .1)=(.0\overline{1},\ .1\overline{0})$
3,1	–1.543689012…	(CLR)L	$\left(\frac{5}{12},\frac{7}{12}\right)$	$(.01\overline{10},\ .10\overline{01})$
6,1	–1.697555393…	$(CLRL^2R)L$	$\left(\frac{41}{96},\frac{55}{96}\right)$	$(.01101\overline{10},\ .10010\overline{01})$
5,2	–1.430357632…	$(CLRL^2)LR$	$\left(\frac{99}{240},\frac{141}{240}\right)$	$(.0110\overline{1001},\ .1001\overline{0110})$
7,3	–1.782233250…	$(CLR^2LRL)LR^2$	$\left(\frac{1799}{4032},\frac{2233}{4032}\right)$	$(.011100\overline{100011},\ .100011\overline{011100})$
3,4	–1.983681352…	$(CLR)R^2L^2$	$\left(\frac{29}{60},\frac{31}{60}\right)$	$(.01\overline{1110},\ .10\overline{0001})$
4,5	–1.874926621…	$(CLR^2)L^3RL$	$\left(\frac{113}{248},\frac{135}{248}\right)$	$(.011\overline{10100},\ .100\overline{01011})$
7,5	–1.995649507…	$(CLR^4L)LR^2LR$	$\left(\frac{975}{1984},\frac{1009}{1984}\right)$	$(.011111\overline{01110},\ .100000\overline{10001})$
8,4	–1.551783943…	$(CLRL^5)L^3R$	$\left(\frac{13605}{32640},\frac{19035}{32640}\right)$	$(.0110101\overline{01011010},\ .1001010\overline{10100101})$
13,4	–1.423729232…	$(CLRL^3\overline{RL}^3R)LRL^2$	$\left(\frac{430845}{1044480},\frac{613635}{1044480}\right)$	$(.011010011001\overline{10010110},\ .100101100110\overline{01101001})$
7,10	–1.996548203…	$(CLR^5)LR^4LR^2L^2$	$\left(\frac{32239}{65472},\frac{33233}{65472}\right)$	$(.011111\overline{1000001110},\ .100000\overline{0111110001})$

Table 2