A Revision of the Lyapunov Exponent in 1D Quadratic Maps

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Only Misiurewicz points strictly have a Lyapunov exponent positive measure in one-dimensional (1D) quadratic maps. Hence, we represent here the Lyapunov exponent, λ , of all the Misiurewicz points which were found and inventoried by us in a former work. As a result of this representation, an infinity of alignments of Misiurewicz points are obtained, and two types of jumps in the values of λ are found.

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The Lyapunov exponent of a one-dimensional (1D) map $x_{k+1} = f(r, x_k)$ is obtained operationally, from Shaw [1] onward, by iterating the map, keeping track of the average natural logarithm of the slope

$$\lambda(r) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \ln\left|f'(r, x_k)\right| . \quad (1)$$

A plot of λ as a function of the parameter for the logistic map $x_{k+1} = rx_k(1 - x_k)$ was presented for the first time by Shaw [1]. The low-order attracting periodic orbits which are relatively wide parameter windows are visible as points where λ is negative [1]. The Lyapunov exponent appears to be positive in relatively wide parameter intervals, which does not agree with the well known fact that in each one of these intervals there are an infinity of attracting periodic orbits with negative λ. That is because if the window width is less than the grid spacing of the computation, the window cannot be detected [1]. Obviously this grid spacing has to be greater than the machine accuracy [2]. In the real Mandelbrot map $x_{k+1} = x_k^2 + c$, see Fig. 1(a), windows with relatively low periods are very narrow. For example, the window with period 39 and symbolic sequence [3] CLRL^{36} located at $c = -1.54368901485464\ldots$ has a width (9×10^{-18}) of the same order of magnitude as the machine accuracy of a personal computer with a 80-bit wordlength (long double precision).

We agree with Holton and May [4] that the numerical computation of λ as a function of the parameter for a 1D quadratic map is, in a strict mathematical sense, nonsense [4]. Quoting Picasso (although here for dynamics not art), the plot is "a lie that tells the truth" [4], and the true version of the plot of the Lyapunov exponent as a function of parameter is thus ultimately not possible to draw [4]. However, the plot can be useful for practical purposes. Even some authors have given analytical expressions of the curve envelope in different zones: close to the Feigenbaum point ($r = 3.5699\ldots$, $c = -1.4011\ldots$) [5], and near to the boundary crisis point ($r = 4$, $c = -2$) [6,7].

In a 1D quadratic map there are three types of points according its parameter values: points where $\lambda < 0$ (stable and superstable periodic points), points where $\lambda = 0$ (bifurcation points), and points where $\lambda > 0$ (Misiurewicz points [8]). We have studied Misiurewicz points in 1D quadratic maps [3,9,10] and, in this letter, we shall study the Lyapunov exponent of these Misiurewicz points. We have seen that only Misiurewicz points are strictly chaotic in the sense that they are always aperiodic when they are iterated with finite machine accuracy (narrow windows are also aperiodic, but they change into periodic when the machine accuracy is improved). According to Jakobson [11] and Farmer [12], chaotic points form a fat Cantor set with positive Lebesgue measure.

Misiurewicz points, $\mathrm{M}_{n,p}$, have a preperiod n and are eventually periodic of period p [9]. Since they are unstable [9], the orbit remains in the cycle during a limited number of iterations when we have finite machine accuracy; then, the orbit leaves the cycle and changes into chaotic. To correctly calculate the Lyapunov exponent of a Misiurewicz point $\mathrm{M}_{n,p}$, Eq. (1) has to be used by only iterating the number of times truly needed. Hence we use the expression

$$\lambda(c) = \frac{1}{p} \sum_{k=n}^{n+p-1} \ln\left|f'(c, x_k)\right| , \quad (2)$$

where we calculate only once the mean value of the information change [1] in the cycle points of the real Mandelbrot map $x_{k+1} = x_k^2 + c$. By using Eq. (2) we have calculated the λ of all the 616 Misiurewicz points inventoried in the Table 1 of Ref. [3], which are shown in Fig. 1(b) (most of them near $c = -2$). A few more points needed for our study are also shown.

As is well known, all the quadratic maps are topologically conjugate [13] and mappings which are topologically conjugate are completely equivalent in terms of their dynamics [14]. In this work we use the real Mandelbrot map. A quadratic map can be "seen" in the real axis neighborhood (the antenna) of the Mandelbrot-like set in its complex form better than in the real line. So, in this work we use the Mandelbrot set antenna to "see" the real Mandelbrot map. Therefore, an interval corresponding to the set of parameter values for which the map has an attracting periodic orbit is then replaced by a cardioid or a disk, and a window of the bifurcation diagram is then replaced by a midget. The cardioid (or the disk) can be named by the symbolic sequence of the representative orbit of the interval, i.e. the superstable periodic orbit that is born from a tangent bifurcation (or a pitchfork bifurcation). The interior crisis point [15] of a window of symbolic sequence *P* is replaced by *tip* (*P*) [16], and the boundary crisis point [15] is replaced by *tip* (C) [16].

Let us note in Fig. 1(b) that a point $[\mathrm{M}_{n,p}, \lambda(\mathrm{M}_{n,p})]$ or $[tip(\mathrm{CLR}), \lambda(tip(\mathrm{CLR}))]$ is represented as $\mathrm{M}_{n,p}$ or $tip(\mathrm{CLR})$ for our own convenience. Likewise, we have added some midgets of the Mandelbrot set antenna in the abscissa axis which, though being complex plane pictures, help us to see better the parameter axis.

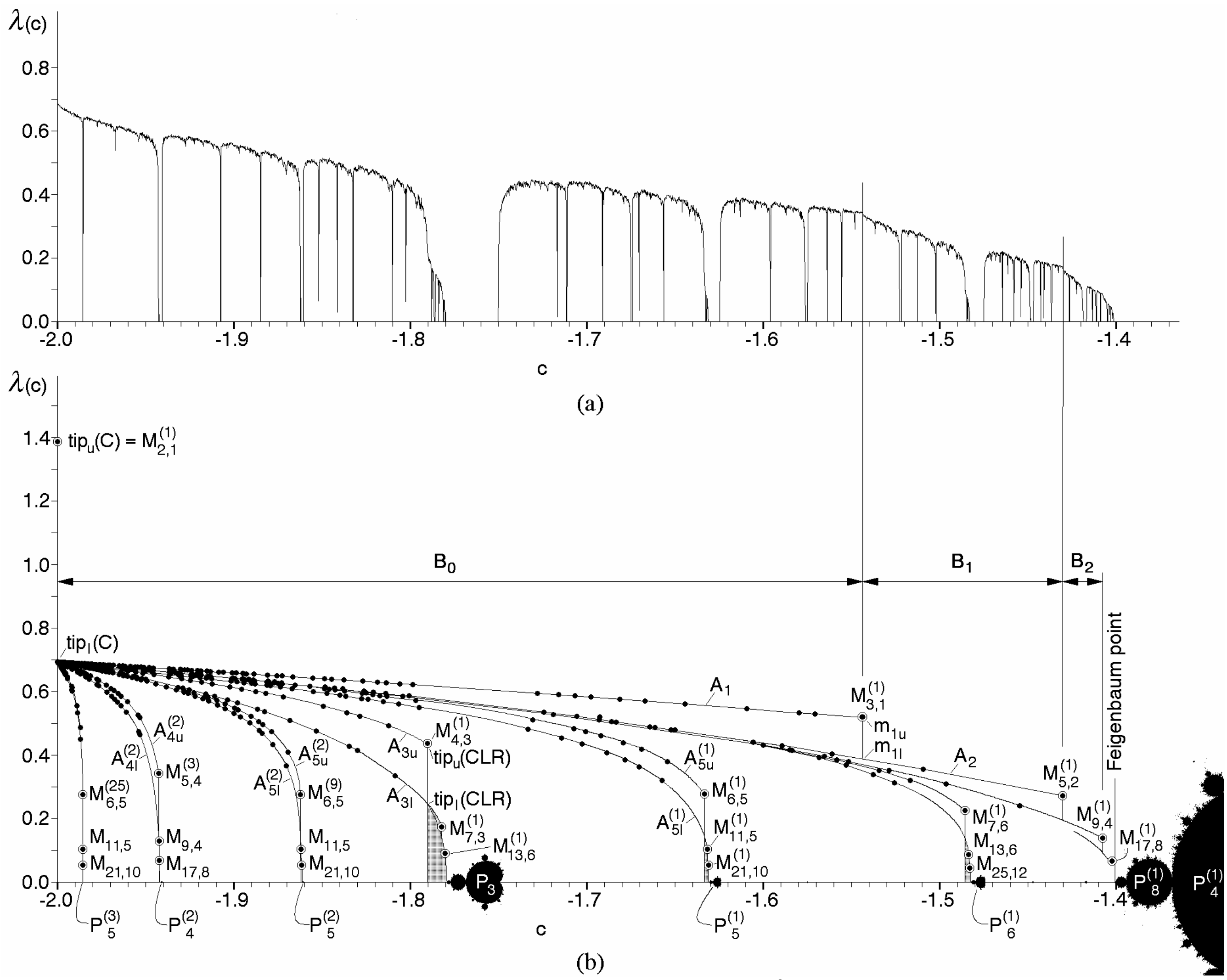


FIG. 1. Lyapunov exponent of the chaotic region of the real Mandelbrot map $x_{k+1} = x_k^2 + c$. (a) Plot of the Lyapunov exponent ($\lambda > 0$) as a function of parameter; (b) Lyapunov exponent of the Misiurewicz points of table 1 of Ref. [3].

First, we note in Fig. 1(b) that the points are in *alignments*. To emphasize alignments, we have joined their points with a thin line. We have the following property:

Property 1. All the Misiurewicz points of the same alignment have the same period.

As can be seen in Fig. 1(b), all the alignments come out from the same point, the tip of the main cardioid whose symbolic sequence is C. We call this point $tip_l(\mathrm{C})$ (we shall see later why this subindex). The alignment which has greater values of λ is $\mathbf{A}_1$, where all the Misiurewicz points have period 1. This alignment finishes in $\mathrm{M}_{3,1}^{(1)}$, the Misiurewicz point that separates the chaotic bands $\mathbf{B}_0$ and $\mathbf{B}_1$ [10]. Next, there comes alignment $\mathbf{A}_2$ where all the Misiurewicz points have period 2. This alignment finishes in $\mathrm{M}_{5,2}^{(1)}$, the separator of the chaotic bands $\mathbf{B}_1$ and $\mathbf{B}_2$. In general, alignment $\mathbf{A}_{2^{i-1}}$ has only Misiurewicz points of period 2^{i-1}, and finishes in the Misiurewicz point $\mathrm{M}_{2^i+1,2^{i-1}}^{(1)}$ [10] which separates the chaotic bands $\mathbf{B}_{i-1}$ and $\mathbf{B}_i$. When $i \to \infty$, the limit alignment finishes in the Feigenbaum point of C. Then, we can associate each one of the alignments to the corresponding separator of chaotic bands. But in the same way, we can associate each one of the alignments to the corresponding hyperbolic component [17] of the period-doubling cascade: $\mathbf{A}_1$ is associated to the main cardioid (period-1), $\mathbf{A}_2$ to the first disk (period-2), and so on.

If we analyze now the rest of the alignments, we observe that two alignments reach each midget of period p, an upper one $\mathbf{A}_{pu}$ and a lower one $\mathbf{A}_{pl}$ (thus, two alignments reach the period-3 midget P_3: $\mathbf{A}_{3u}$ and $\mathbf{A}_{3l}$). The alignments corresponding to midgets with the same period can be distinguished by indicating the order of appearance. Thus, the two alignments of the first of the three period-5 midgets, $\mathrm{P}_5^{(1)}$ are $\mathbf{A}_{5u}^{(1)}$ and $\mathbf{A}_{5l}^{(1)}$.Then, we can enunciate the following property:

Property 2. Each disk is associated to one alignment, and each cardioid is associated to two alignments (except for the cardioid C which only has one).

Let $G(p)$ be the number of hyperbolic components (cardioids and disks) of period p, that is well known [18-20]. The number of cardioids is $C(p) = G(p) - G(p/2)$ when p is even, and $C(p) = G(p)$ when p is odd [19]. The number of disks is $D(p) = G(p/2)$ when p is even, and $D(p) = 0$ when p is odd [19]. Hence, by taking into account the property 2, the number of alignments is $A(p) = 2G(p) - G(p/2)$ when p is even, and $A(p) = 2G(p)$ when p is odd (see Table 1).

Table 1. Number of alignments of Misiurewicz points of periods $p \le 11$.

p	1	2	3	4	5	6	7	8	9	10	11
$A(p)$	1	1	2	3	6	9	18	30	56	99	186

It is clear that the Misiurewicz points number of period p is much greater than the hyperbolic components number of the same period, since each hyperbolic component is associated to one or two alignments and each alignment has an infinity of Misiurewicz points.

The alignments we have just described correspond to the first generation components. The second generation components have their own alignments (see the shaded zone in the period-3 midget; the Lyapunov exponent values are the same, except for a scale factor, that those we have just seen). That is, from $tip_l(\mathrm{CLR})$ an infinity of alignments come out, one to each period-doubling cascade component of the midget CLR, and two to each secondary midget of CLR (we can imagine that the shaded zone is enlarged to the total Fig. 1 (b)).

We can start from the point $tip_l(\mathrm{C})$ and reach the Feigenbaum point of C by proceeding along the alignments $\mathbf{A}_1$, $\mathbf{A}_2$,…, along a characteristic stairway which has the following property:

Property 3. The characteristic stairway is an upper bound of the Lyapunov exponent. All the Misiurewicz points whose Lyapunov exponents are in the characteristic stairway are characteristic Misiurewicz points [9].

Indeed, it is obviously an upper bound (we shall add afterwards a point, $tip_u(\mathrm{C})$). Likewise it is clear that the characteristic stairway has all the Lyapunov exponents of the characteristic Misiurewicz points and only them. Indeed, characteristic Misiurewicz points in $\mathbf{B}_0$ are $\mathrm{M}_{n,1}$, which constitute $\mathbf{A}_1$. The only $\mathrm{M}_{n,2}$ points which are characteristic Misiurewicz points are in $\mathbf{B}_1$, and they constitute the stairway flight of $\mathbf{A}_2$. Likewise we can see that the only $\mathrm{M}_{n,2^i}$ which are characteristic Misiurewicz points are in $\mathbf{B}_i$ and they constitute the stairway flight of $\mathbf{A}_{2^i}$.

For the logistic map, Crutchfield, Farmer and Huberman [21] observed a "peculiar upturn" in $\lambda(r)$ toward ln 2 near $r = 4$, and also saw "kinks" in $\lambda(r)$ where chaotic bands merge. As can be seen in Fig. 1(b), these changes can occur due to a *jump* of λ associated to some Misiurewicz points. In these points, λ has a double value: an upper one, λ_u, and a lower one, λ_l. Let us see the following property:

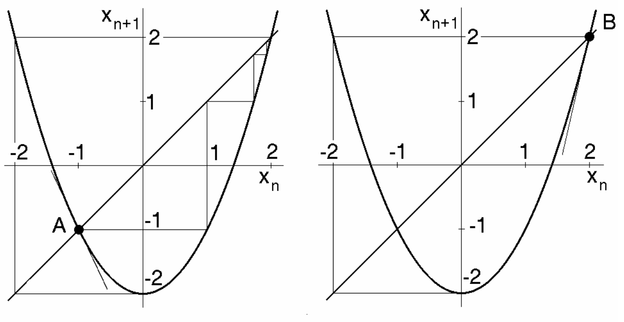
Property 4. There exist jumps of the Lyapunov exponent in the tip of the antenna of every midget (*tip jumps*), and in the separators of chaotic bands (*separator jumps*).

As can be seen in Fig. 1(b), there is an isolated upper point at $c = -2$, $tip_u(\mathrm{C})$, and also a lower point already known, $tip_l(\mathrm{C})$, from where all the alignments of the main cardioid C come out. We reach $tip_u(\mathrm{C})$ through an infinity of hyperbolic components: $tip_u(\mathrm{C}) = (\mathrm{CL})\mathrm{R} = \mathrm{M}_{2,1}^{(1)}$ [16]. However, as can be seen in Fig. 1(b), to reach $tip_l(\mathrm{C})$ we can go through an infinity of Misiurewicz points alignments. In $\mathbf{A}_1$ we go through $\mathrm{m}_{0,\alpha} = \mathrm{M}_{\alpha+2,1}^{(1)} = (\mathrm{CLR}^{a})\mathrm{L}$, $\alpha = 1,2,3\ldots$ [9]. When $\alpha \to \infty$ we have $(\mathrm{CLR}^{\infty})\mathrm{L} = (\mathrm{CL})\mathrm{R}$. Hence, in the infinity $tip_l(\mathrm{C})$ jumps to $tip_u(\mathrm{C})$. The same type of jump can be observed in the tip of the antenna of every midget.

Thus, let us consider $tip_l(\mathrm{CLR})$ and $tip_u(\mathrm{CLR})$ in Fig. 1(b). We reach $tip_u(\mathrm{CLR})$ through an infinity of hyperbolic components (from the right): $tip_u(\mathrm{CLR}) = (\mathrm{CLR}^2)\mathrm{LRL}$ [16] and, as can be seen in Fig. 1(b), we can also reach it (from the left) through the alignment $\mathbf{A}_{3u}$. This was impossible in the case of the main cardioid tip because there was no alignment on the left. We can reach $tip_l(\mathrm{CLR})$ through an infinity of alignments from the right and, in addition, through the alignment $\mathbf{A}_{3l}$ from the left. If we choose the alignment $\mathbf{A}_3(\mathrm{CLR})$ (which corresponds to $\mathbf{A}_1(\mathrm{C})$ in the case of $tip_l(\mathrm{C})$), we reach $tip_l(\mathrm{CLR})$ through the Misiurewicz points $\mathrm{M}_{3\alpha+4,3}^{(1)}$ ($\alpha = 1,2,3\ldots$) of symbolic sequences $(\mathrm{CLR}^2\mathrm{LRL})\mathrm{LR}^2$, $(\mathrm{CLR}^2\,\overline{\mathrm{LRL}}^2)\mathrm{LR}^2$, …, $(\mathrm{CLR}^2\,\overline{\mathrm{LRL}}^{\alpha})\mathrm{LR}^2$, … and again, when $\alpha \to \infty$, $tip_l(\mathrm{CLR})$ jumps to $tip_u(\mathrm{CLR})$.

It is possible to obtain the Lyapunov exponent of the map $x_{k+1} = x_k^2 + c$ for $c = -2$ in an easy *graphical way*. Other authors have obtained it analytically, in a beautiful but complex way [22]. We apply the graphical iteration method due to Feigenbaum [23] to determine the successive iterates of the corresponding Misiurewicz point, and we calculate the exponent by means of Eq. (2) in the two following situations:

a) Very close to $c = -2$, which corresponds to $tip_l(\mathrm{C})$, we have some Misiurewicz points as $(\mathrm{CLR}^{\alpha})\mathrm{L}$ where the orbit is eventually the unstable fixed point A of Fig. 2(a). Thus, we have $\mathrm{M}_{9,1}^{(11)} = (\mathrm{CLR}^7)\mathrm{L}$ in $c = -1.999899589\ldots$ [3]. In the limit, the map is $x_{k+1} = x_k^2 - 2$ and A is in $x_\mathrm{A} = -1$. The slope in this point is -2 and, by applying Eq. (2), $\lambda[tip_l(\mathrm{C})] = \ln 2$.



(a) (b)

FIG. 2. Graphical iteration in the $tip(\mathrm{C})$ of the map $x_{k+1} = x_k^2 + c$. (a) $\mathrm{M}_{9,1}^{(11)} = (\mathrm{CLR}^7)\mathrm{L}$ ($c = -1.999899589\ldots$); (b) $\mathrm{M}_{2,1}^{(1)} = (\mathrm{CL})\mathrm{R}$ ($c = -2$).

b) In $c = -2$, that corresponds to $tip_u(\mathrm{C})$, we have the Misiurewicz point $(\mathrm{CL})\mathrm{R}$, and the orbit finishes in the unstable fixed point B of Fig. 2(b). Now the map is $x_{k+1} = x_k^2 - 2$ and B is in $x_\mathrm{B} = 2$. The slope in this point is 4 and, by applying Eq. (2), $\lambda[tip_u(\mathrm{C})] = \ln 4 = 2\ \ln\ 2$.

Therefore, in *tip*(C) the Lyapunov exponent jumps $\ln\ 2$ (from $\ln\ 2$ to $2\ \ln\ 2$). These jumps are present in the remaining tips (they are obvious in Fig. 1(a)). Thus, we have $\lambda\left[tip_l(\mathrm{CLR})\right] = 0.2372\ldots$ and $\lambda\left[tip_u(\mathrm{CLR})\right] = 0.4374\ldots$

Let us see now the separator jumps. As we already mentioned, the alignments $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_4$, … come out from the $tip_l(\mathrm{C})$ and finish in the separators of chaotic bands. In these separators there are also a jump in the value of the Lyapunov exponent but, now, the jump is not so evident in Fig. 1(a). In every chaotic band, the upper alignment is clearly separated from the others (an infinity number of non-coincidental ones). If an alignment disappears, it has little influence in the value of λ. Then, presumably, its disappearance causes a change in the slope of the curve envelope in Fig. 1(a), instead of a jump.

However, these separator jumps are very obvious in Fig. 1(b). There is only one upper point, m_{iu}, where $i = 1,2,4,8,\ldots$ [9] and there are an infinity of lower points. Let m_{il} be the upper one of these lower points. We have $\mathrm{m}_{1u} = \mathrm{M}_{3,1}^{(1)}$ [9] and $\mathrm{M}_{3,1}^{(1)} = (\mathrm{CLR})\mathrm{L}$ [24] (we reach the same result through the alignment $\mathbf{A}_1$ from the left). Otherwise, m_{1l}, which is in the alignment $\mathbf{A}_2$, is the limit of Misiurewicz points $(\mathrm{CLRL}^{\alpha})\mathrm{LR}$. We obtain this value in both cases, when we reach it from the left through non-characteristic Misiurewicz points (then α is odd), and from the right through characteristic Misiurewicz points (then α is even). When $\alpha \to \infty$, $(\mathrm{CLRL}^{\alpha})\mathrm{LR}$ turns to $(\mathrm{CLR})\mathrm{L}$, i.e. m_{1l} jumps to m_{1u}.

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